Bunched implication logic is undecidable

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BI-algebras: algebraic semantics

A *BI-algebra* is an algebra of the form $\mathbf{A} = (A, \wedge, \vee, \to, \top, \bot, \cdot, -\!\!*, 1)$ where

- $(A, \wedge, \vee, \top, \bot)$ is a bounded lattice, and
- for all $x, y, z \in A$,

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BI algebras form algebraic semantics for *bunched implication logic*.

Bunched implication logic: the calculus

$$\dfrac{}{\phi \Rightarrow \phi}\ \mathsf{ax} \qquad \dfrac{\Gamma(\Delta) \Rightarrow \phi}{\Gamma(\Delta;\Delta') \Rightarrow \phi}\ \mathsf{w} \qquad \dfrac{\Gamma(\Delta;\Delta) \Rightarrow \phi}{\Gamma(\Delta) \Rightarrow \phi}\ \mathsf{c}$$

$$\dfrac{\Delta \Rightarrow \phi \qquad \Gamma(\phi) \Rightarrow \psi}{\Gamma(\Delta) \Rightarrow \psi}\ \mathsf{cut} \qquad \dfrac{\Gamma \Rightarrow \phi}{\Delta \Rightarrow \phi}\ \Delta \equiv \Gamma$$

$$\dfrac{\Gamma(\phi,\psi) \Rightarrow \chi}{\Gamma(\phi \cdot \psi) \Rightarrow \chi}\ \cdot_L \qquad \dfrac{\Gamma \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \phi \cdot \psi}\ \cdot_R$$

$$\dfrac{\Delta \Rightarrow \phi \qquad \Gamma(\psi) \Rightarrow \chi}{\Gamma(\Delta,\ \phi \mathrel{-\!\!*} \psi) \Rightarrow \chi}\ \mathrel{-\!\!*}_L \qquad \dfrac{\Gamma,\phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \mathrel{-\!\!*} \psi}\ \mathrel{-\!\!*}_R$$

$$\dfrac{\Gamma \Rightarrow \phi_i\ (i=1,2)}{\Gamma \Rightarrow \phi_1 \vee \phi_2}\ \vee_{Ri} \qquad \dfrac{\Gamma(\phi) \Rightarrow \chi \qquad \Gamma(\psi) \Rightarrow \chi}{\Gamma(\phi \vee \psi) \Rightarrow \chi}\ \vee_L$$

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$$\dfrac{\Delta \Rightarrow \phi \qquad \Gamma(\psi) \Rightarrow \chi}{\Gamma(\Delta;\ \phi \to \psi) \Rightarrow \chi}\ \to_L \qquad \dfrac{\Gamma;\phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi}\ \to_R$$

$$\dfrac{\Gamma(\varnothing_\times) \Rightarrow \phi}{\Gamma(1) \Rightarrow \phi}\ 1_L \qquad \dfrac{}{\varnothing_\times \Rightarrow 1}\ 1_R$$

$$\dfrac{}{\Gamma(\bot) \Rightarrow \phi}\ \bot_L \qquad \dfrac{\Gamma(\varnothing_+) \Rightarrow \phi}{\Gamma(\top) \Rightarrow \phi}\ \top_L \qquad \dfrac{}{\varnothing_+ \Rightarrow \top}\ \top_R$$

Separation logic and Bunched Implication Logic

Bunched implication (BI) logic (introduced by D. Pym) is used to study pointer structures, memory allocation and concurrent programming. It serves as the Hoare logic for separation logic.

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BI-logic is a substructural logic and it combines an additive/Heyting implication (*the logical part*) and a multiplicative/linear implication (*the dynamic part*).

It has a nice proof theory (an analytic calculus with cut elimination).

And-branching counter machines

An *and-branching counter machine (ACM)* is a *finite* structure $\mathtt{M} = (Q, q_f, R, I)$, where Q is a set of *states*, $q_f \in Q$ is the final state, $R = \{r_1, \ldots, r_k\}$ is the set of *register tokens*, and I is a set of instructions of the following forms, where $q, q', q_1, q_2 \in Q$ and $r \in R$:

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Fact: There exists an ACM $\mathtt{M}_{\mathsf{Und}}$ such that it is undecidable: given a configuration c of $\mathtt{M}_{\mathsf{Und}}$, does $c \rightsquigarrow q_f \vee \cdots \vee q_f$ hold in $\mathtt{M}_{\mathsf{Und}}$? We write $\overline{q_f}$ for $q_f \vee \cdots \vee q_f$.

Soundness

Given an ACM $\mathtt{M} = (Q, q_f, R, I)$ we define the terms:

$$i := \bigwedge I, \qquad t := (i \mathrel{-\!\!*} q_f) \to q_f, \qquad \theta := (\top \mathrel{-\!\!*} t) \wedge 1.$$

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Theorem 1. If $c \rightsquigarrow \overline{q_f}$ then $\mathsf{BI} \models c\theta \leq q_f$, for every ACM $\mathtt{M}$ and configuration c.

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Proof. (By induction on the length of the computation.)
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hence $c\theta \leq (i \mathrel{-\!\!*} c')\theta \leq i \mathrel{-\!\!*} c'\theta$. So, [using $x \wedge (x \to q_f) \leq q_f$]

$$c\theta = c\theta \wedge c\theta \leq (i \mathrel{-\!\!*} c'\theta) \wedge \top(\top \mathrel{-\!\!*} t) \leq (i \mathrel{-\!\!*} q_f) \wedge t = (i \mathrel{-\!\!*} q_f) \wedge [(i \mathrel{-\!\!*} q_f) \to q_f] \leq q_f.$$

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Corollary. The equational theory of BI is undecidable.

Thank you!

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Lemma. For every ACM $\mathtt{M}$, $\boldsymbol{W}_{\mathtt{M}}$ is a commutative distributive residuated frame.

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$x \backslash\!\!\backslash u := u(x \circ \_), \; u /\!\!/ y := u(\_ \circ y), \; x \searrow u := u(x \curlywedge \_), \; u \swarrow y := u(\_ \curlywedge y)$.

Lemma. For every ACM $\mathtt{M}$, $\boldsymbol{W}_{\mathtt{M}}$ is a commutative distributive residuated frame.

Proof. We first check the two nuclear properties. We have $x \curlywedge y \; N \; u$ iff $u(x \curlywedge y) \preccurlyeq q_f$ iff $x \; N \; u(\_ \curlywedge y) = u \swarrow y$. Likewise, for $\searrow$, $\backslash\!\!\backslash$, and $/\!\!/$.

Residuated frames

The operations $\cdot, 1, \wedge, \top$ of M are often written as $\circ, \varepsilon, \curlywedge, \epsilon$.

On K we also define the operations $\circ, \varepsilon, \curlywedge, \epsilon$ by

$$(\bigvee x_i) \circ (\bigvee y_j) = \bigvee (x_i \circ y_j) \text{ and } (\bigvee x_i) \curlywedge (\bigvee y_j) = \bigvee (x_i \curlywedge y_j),$$

for all $x_i, y_j \in M$, and the constants of M; this yields the algebra $\boldsymbol{K} := (K, \circ, \varepsilon, \curlywedge, \epsilon, \vee)$.

We define $\boldsymbol{W}_{\mathtt{M}} := (M, S_M, N, \circ, \varepsilon, \backslash\!\backslash, /\!/, \curlywedge, \epsilon, \leftthreetimes, \rightthreetimes)$, where

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Next, we check the four frame structural properties. If $x \curlywedge y \; N \; u$ then $u(x \curlywedge y) \preccurlyeq q_f$ so $u(y \curlywedge x) \preccurlyeq u(x \curlywedge y) \preccurlyeq q_f$, hence $y \curlywedge x \; N \; u$. Likewise for exchange for $\circ$. If $x \curlywedge x \; N \; u$, then $u(x) \preccurlyeq u(x \curlywedge x) \preccurlyeq q_f$, so $x \; N \; u$. If $x \; N \; u$, then $u(x \curlywedge y) \preccurlyeq u(x) \preccurlyeq q_f$, so $x \curlywedge y \; N \; u$.

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To prove $1 \subseteq e(i)$, note that: $1 \subseteq e(i) = \bigcap(\gamma(d_i) \multimap \gamma(d_i'))$ iff $1 \subseteq \gamma(d_i) \multimap \gamma(d_i')$ for all $d \multimap d' \in I$ iff $\gamma(d_i) \subseteq \gamma(d_i')$ iff $d \in \gamma(d')$ for all $d \multimap d' \in I$. So, for a given $d \multimap d' \in I$, we assume $\gamma(d') \subseteq u^{\lhd}$, i.e., $d' \in u^{\lhd}$, and will show $d \in u^{\lhd}$; we distinguish cases.

If d' is a configuration, then $d' \in u^{\lhd}$ gives $u(d') \preccurlyeq \overline{q_f}$, while $d \multimap d' \in I$ gives $d \rightsquigarrow d'$, so $d \preccurlyeq d'$, hence $u(d) \preccurlyeq u(d') \preccurlyeq \overline{q_f}$; thus $d \in u^{\lhd}$.

Completeness

Theorem 2. If BI $\models c\theta \leq q_f$ then $c \preccurlyeq \overline{q_f}$, for every ACM $\mathtt{M}$ and configuration c.

Proof. We have $\boldsymbol{W}_{\mathtt{M}}^{+}, e \models c\theta \leq q_f$ where e extends the assignment $e(x) = \gamma(\{x\})$ for $x \in Q \cup R$ and $\gamma(X) = X^{\rhd\lhd}$. So, $e(c\theta) \subseteq e(q_f) = \gamma(\{q_f\})$.
Since $\varepsilon \circ q_f \preccurlyeq q_f$, we get $q_f\ N\ u_\varepsilon$, where $u_\varepsilon := (\varepsilon \circ \_)$, i.e., $q_f \in u_\varepsilon^{\lhd}$, so $\gamma(\{q_f\}) \subseteq u_\varepsilon^{\lhd}$.
Thus, $e(c) \circ_\gamma e(\theta) = e(c\theta) \subseteq u_\varepsilon^{\lhd}$. We will prove that $e(c) = \gamma(\{c\})$ and $1_{\boldsymbol{W}_{\mathtt{M}}^{+}} \subseteq e(i)$.

To prove $e(c) = \gamma(c)[:= \gamma(\{c\})]$, let $c = a_1 \cdot a_2 \cdots a_m$, for some $a_i \in Q \cup R$. Then
$e(c) = e(a_1 \cdot a_2 \cdots a_m) = e(a_1) \circ_\gamma e(a_2) \circ_\gamma \cdots \circ_\gamma e(a_m) =$
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To prove $1 \subseteq e(i)$, note that: $1 \subseteq e(i) = \bigcap(\gamma(d_i) \mathrel{-\!\!*} \gamma(d_i'))$ iff $1 \subseteq \gamma(d_i) \mathrel{-\!\!*} \gamma(d_i')$ for all $d \mathrel{-\!\!*} d' \in I$ iff $\gamma(d_i) \subseteq \gamma(d_i')$ iff $d \in \gamma(d')$ for all $d \mathrel{-\!\!*} d' \in I$. So, for a given $d \mathrel{-\!\!*} d' \in I$, we assume $\gamma(d') \subseteq u^{\lhd}$, i.e., $d' \in u^{\lhd}$, and will show $d \in u^{\lhd}$; we distinguish cases.

If d' is a configuration, then $d' \in u^{\lhd}$ gives $u(d') \preccurlyeq \overline{q_f}$, while $d \mathrel{-\!\!*} d' \in I$ gives $d \rightsquigarrow d'$, so $d \preccurlyeq d'$, hence $u(d) \preccurlyeq u(d') \preccurlyeq \overline{q_f}$; thus $d \in u^{\lhd}$.

If $d' = q_1 \vee q_2$, then $d' \in u^{\lhd}$ gives $u(q_1) \preccurlyeq \overline{q_f}$ and $u(q_2) \preccurlyeq \overline{q_f}$, while $d \mathrel{-\!\!*} d' \in I$ gives $d \rightsquigarrow q_1 \vee q_2$, so $d \preccurlyeq q_1 \vee q_2$, hence
$u(d) \preccurlyeq u^{\boldsymbol{J}}(q_1 \vee q_2) = u^{\boldsymbol{J}}(q_1) \vee u^{\boldsymbol{J}}(q_2) \preccurlyeq \overline{q_f} \vee \overline{q_f} \preccurlyeq \overline{q_f}$; thus $d \in u^{\lhd}$.

Cont.

$$i := \bigwedge I, \qquad t := (i \mathrel{-\!\!*} q_f) \to q_f, \qquad \theta := (\top \mathrel{-\!\!*} t) \wedge 1.$$

Theorem 2. If BI $\models c\theta \leq q_f$ then $c \preccurlyeq \overline{q_f}$, for every ACM $\mathtt{M}$ and configuration c.

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Proof. (cont) Using $1 \subseteq e(i)$, we prove $e(\theta) = 1$. Indeed,
$e(t) = [e(i) \mathrel{-\!\!*} \gamma(q_f)] \to \gamma(q_f) \supseteq [1 \mathrel{-\!\!*} \gamma(q_f)] \to \gamma(q_f) = \gamma(q_f) \to \gamma(q_f) = \top_{\boldsymbol{W}_{\mathtt{M}}^+}$, so
$1 \supseteq e(\theta) = [e(\top) \mathrel{-\!\!*} e(t)] \cap 1 \supseteq [e(\top) \mathrel{-\!\!*} \top_{\boldsymbol{W}_{\mathtt{M}}^+}] \cap 1 \supseteq 1.$

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$1 \supseteq e(\theta) = [e(\top) \mathrel{-\!\!*} e(t)] \cap 1 \supseteq [e(\top) \mathrel{-\!\!*} \top_{\boldsymbol{W}_{\mathtt{M}}^{+}}] \cap 1 \supseteq 1.$

Now, from $e(c) = \gamma(c)$, $e(\theta) = 1$ and $e(c) \circ_\gamma e(\theta) \subseteq u_\varepsilon^{\lhd}$, we get
$c \in \gamma(c) = e(c) \circ_\gamma e(\theta) \subseteq u_\varepsilon^{\lhd}$, i.e., $c \preccurlyeq \varepsilon \circ c = u_\varepsilon(c) \preccurlyeq \overline{q_f}$, hence $c \preccurlyeq \overline{q_f}$.

Completeness Part II

Theorem 3. If $c \preccurlyeq \overline{q_f}$ then $c \rightsquigarrow \overline{q_f}$, for every ACM $\mathtt{M}$ and configuration c.

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Proof. Claim: If $z \preccurlyeq_{n+1} q_f$ and z contains $\curlywedge$, then there exists z' such that $z \preccurlyeq_1 z' \preccurlyeq_n q_f$ and the first step was weakening.

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If we accept the Claim, in a given computation witnessing $c \preccurlyeq \overline{q_f}$ there might be some instruction steps first, applied successively to c, and then the first application (if any) of contraction along a branch giving $c' \preccurlyeq c' \curlywedge c'$, where c' is a configuration, or more generally $c'(x) \preccurlyeq c'(x \curlywedge x)$, where x is a $\circ$-term.

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To prove the Claim, we distinguish cases for what the first step in $z \preccurlyeq_{n+1} q_f$ is. If $v(x) \preccurlyeq_1 v(x \curlywedge x) \preccurlyeq_n q_f$, then by IH $v(x \curlywedge x) \preccurlyeq_1 v(x) \preccurlyeq_{n-1} q_f$, in which case $v(x) \preccurlyeq_{n-1} q_f$, or $v(x \curlywedge x) \preccurlyeq_1 v'(x \curlywedge x) \preccurlyeq_{n-1} q_f$, in which case $v(x) \preccurlyeq_1 v'(x) \preccurlyeq_1 v'(x \curlywedge x) \preccurlyeq_{n-1} q_f$. The remaining cases of non-instruction steps are similar.

Cont.

To discuss the instruction steps we introduce some terminology. For $z \in M$ and a leaf d of the structure tree of z, we denote by $m(d)$ the biggest principal donwset of the structure tree of z that contains d and does not contain any $\curlywedge$; also we denote by $\curlywedge_d$ be the occurrence of $\curlywedge$ right above $m(d)$, if any. We say: $z = v(m(d))$ is in *$\circ$-form* w.r.t. d.

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In a step $u(x \curlywedge y) \preccurlyeq_1 u(x)$ we say that the weakening was *applied* to the displayed $\curlywedge$ and that y was *removed*; if $v(y) = u(x \curlywedge y)$, then we set $\overline{v} := u(x)$. Therefore, if weakening was applied in $v(m(d))$, then either it was applied to $\curlywedge_d$ and $m(d)$ was removed, i.e., $v(m(d)) \preccurlyeq \overline{v}$, or it was applied to some $\curlywedge$ inside v (possibly to $\curlywedge_d$ even) but $m(d)$ was not removed, i.e., $v(m(d)) \preccurlyeq v'(m(d))$, for some v'.
If $v(m(q)) \preccurlyeq_1 v(m(q_1)) \vee v(m(q_2)) \preccurlyeq_n q_f$, where the expressions are in $\circ$-form, then by IH either $v(m(q_i)) \preccurlyeq_1 v'(m(q_i)) \preccurlyeq_{n-1} q_f$ for both $i \in \{1, 2\}$ (neither one of $m(q_1)$ and $m(q_2)$ was removed by the weakening), in which case
$v(m(q)) \preccurlyeq_1 v'(m(q)) \preccurlyeq_1 v'(m(q_1)) \vee v'(m(q_2)) \preccurlyeq_{n-1} q_f$, or $v(m(q_i)) \preccurlyeq_1 \overline{v} \preccurlyeq_{n-1} q_f$ for some $i \in \{1, 2\}$ (at least one of $m(q_1)$ and $m(q_2)$ was removed by the weakening), in which case we have $v(m(q)) \preccurlyeq_1 \overline{v} \preccurlyeq_{n-1} q_f$.
If $v(m(qr)) \preccurlyeq_1 v(m(q')) \preccurlyeq_n q_f$, where the expressions are in $\circ$-form, then by IH either $v(m(q')) \preccurlyeq_1 v'(m(q')) \preccurlyeq_{n-1} q_f$, in which case
$v(m(qr)) \preccurlyeq_1 v'(m(qr)) \preccurlyeq_1 v'(m(q')) \preccurlyeq_{n-1} q_f$, or $v(m(q')) \preccurlyeq_1 \overline{v} \preccurlyeq_{n-1} q_f$, in which case we have $v(m(q)) \preccurlyeq_1 \overline{v} \preccurlyeq_{n-1} q_f$.