Revisiting the localic Kuratowski-Mrówka theorem

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Overview

1. **Notation (and some history)**
2. **Constructing binary coproducts of frames**
3. **Characterising compactness using binary coproducts and closed maps**

Notation (and some history)

Some notation and definitions

- Any meet-semilattice (abbreviated MSL) L will always have a top 1.
- A frame[1] is a complete lattice L in which, given any $x \in L$ and $S \subseteq L$, $x \wedge (\bigvee S) = \bigvee\{x \wedge s \mid s \in S\}$.

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- Let **Top** denote the category of topological spaces and continuous maps. There is a natural functor $\Omega : \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$ defined by taking each space X and sending it to its poset of open sets ΩX, and each continuous map $X \xrightarrow{f} Y$ to the induced preimage map $f^{-1} : \Omega Y \to \Omega X$.

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- Thus binary coproducts in **Frm** are the analogous construction to binary products in **Top**.

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- [Mrówka, 1959] proves that, given a topological space X, if the projection $p_2 : X \times Y \to Y$ is closed for all topological spaces Y, then X is compact.

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- [Papert, 1959] provides the first construction of coproducts in **Frm** – currently not publicly available.
- [Dowker and Strauss, 1977] another construction of coproducts in **Frm** – proves that a coproduct of compact frames is compact.

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- [Pultr and Tozzi, 1992] establishes that, for a given frame L_1, if the injection $L_2 \to L_1 \oplus L_2$ is closed for all frames L_2, then L_1 is compact.

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Thus we have:

The frame L_1 is compact $\iff$ for all frames L_2 the injection $L_2 \to L_1 \oplus L_2$ is closed.

The "localic/pointfree Kuratowski-Mrówka theorem"

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In [Pultr and Tozzi, 1992], the following lemma is presented:

Lemma

Let A be a compact frame, and B a nontrivial frame. Put $1_{A\oplus B} = \bigvee_{i\in I}(a_i \oplus b_i)$. Then there exists a finite $K \subseteq I$ such that $\bigvee_{i\in K} a_i = 1$ and $\bigwedge_{i\in K} b_i \neq 0$.

For the proof, the reader is referred to [Kříž and Pultr, 1989, Theorem 3.9] – the proof of which uses the axiom of choice.

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Our goal is to show that the above lemma can be proved directly without using the axiom of choice. We will then use this lemma to construct the Kuratowski-Mrówka theorem for frames (choice-free).

Constructing binary coproducts of frames

Free sup-lattices (on preordered sets)

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Theorem

For any preordered set A there is a complete lattice L and an order-preserving map $h : A \to L$ satisfying the following universal property: for any complete lattice M and any order-preserving map $f : A \to M$, there is a unique all-join-preserving map $g : L \to M$ for which $f = gh$.

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Sketch of proof.

For a preordered set A put $L = \mathcal{D}A$, and $h = \eta_A : a \mapsto {\downarrow}a$. Let M be a complete lattice, and $f : A \to M$ an order-preserving map. Define $g : \mathcal{D}A \to M$ by $g(X) = \bigvee f[X]$. Then g satisfies the conditions of the theorem. □

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The pair $(\mathcal{D}A, \eta_A)$ is called the downset completion of A.

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$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & \mathcal{D}A \arrow[d, dashed, "\exists!\text{frame map } g"] \\ & M. \end{tikzcd}$$

$\square$

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For MSLs A_1 and A_2, there is a frame L and MSL-morphisms $A_1 \xrightarrow{q_1} L \xleftarrow{q_2} A_2$ satisfying the following universal property: for any frame M and MSL-morphisms $A_1 \xrightarrow{f_1} M \xleftarrow{f_2} A_2$ there is a unique frame homomorphism $g : L \to M$ for which $\forall i$ $f_i = gq_i$.

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Proof idea.

Put $L = \mathcal{D}(A_1 \times A_2)$ and $\forall i$ define $q_i : A_i \to \mathcal{D}(A_1 \times A_2)$ by $q_i(x) = \{y \in A_1 \times A_2 \mid y_i \leq x\}$; i.e. $q_1(x) = {\downarrow}(x, 1)$ and $q_2(w) = {\downarrow}(1, w)$.

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Note! If L_1 and L_2 are frames then the MSL-morphisms $L_1 \xrightarrow{q_1} \mathcal{D}(L_1 \times L_2) \xleftarrow{q_2} L_2$ are not frame homomorphisms.

We let L_1 and L_2 be frames.

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Propostion

$C\text{-Idl}(L_1 \times L_2)$ is a sublocale of $\mathcal{D}(L_1 \times L_2)$.

Define maps $L_1 \xrightarrow{q_1} C\text{-Idl}(L_1 \times L_2) \xleftarrow{q_2} L_2$ by $q_i(x) = \{y \in L_1 \times L_2 \mid y_i \leq x\}$.

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Proof idea.

Let M be a frame and consider frame homomorphisms $h_i : L_i \to M$. Define a map $g : C\text{-Idl}(L_1 \times L_2) \to M$ by $g(X) = \bigvee\{h_1(a) \wedge h_2(b) \mid a \in X_1, b \in X_2\}$. Then g is the unique frame homomorphism $C\text{-Idl}(L_1 \times L_2) \to M$ for which $\forall i \; h_i = gq_i$. $\square$

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Note! If one of the frames L_1 or L_2 is trivial, then their coproduct in **Frm** is trivial.

We will now refer to the frames L_1 and L_2 as: $L_1 = A$, and $L_2 = B$. We will denote their coproduct in **Frm** by $A \xrightarrow{q_A} A \oplus B \xleftarrow{q_B} B$, or simply $A \oplus B$.

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Lemma

For nontrivial frames A and B, if $1_{A \oplus B} = \bigvee_{i \in I}(q_A(a_i) \wedge q_B(b_i))$, then $1_A = \bigvee_{i \in I} a_i$ and $1_B = \bigvee_{i \in I} b_i$.

Characterising compactness using binary coproducts and closed maps

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A cover of a frame A is a subset $U \subseteq A$ such that $\bigvee U = 1$. If U is a cover of A, then a subcover of U is a subset $W \subseteq U$ such that $\bigvee W = 1$.

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Example

A topological space X is compact iff its frame of opens ΩX is compact.

The Kříž-Pultr lemma

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For any compact frame A and any nontrivial frame B, if $1_{A\oplus B} = \bigvee_{i\in I}(q_A(a_i) \wedge q_B(b_i))$, then there exists a finite $K \subseteq I$ such that $1_A = \bigvee_{i\in K} a_i$ and $0_B \neq \bigwedge_{i\in K} b_i$.

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Sketch of proof.

(Note that if A is trivial, then take $K = \emptyset$). Suppose A is nontrivial and compact, B nontrivial.

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Definition

A frame A satisfies the *unit-decomposition property* (UD) iff for any frame B and each decomposition $1_{A\oplus B} = \bigvee_{i\in I}(q_A(a_i) \wedge q_B(b_i))$, the set $\{\bigwedge_{i\in K} b_i \mid K \subseteq I, \bigvee_{i\in K} a_i = 1\}$ is a cover of B.

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Closed frame homomorphisms

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A frame homomorphism $h : B \to A$ is closed iff for each $a \in A$ and $b, b' \in B$, if $h(b') \leq a \vee h(b)$, then $b' \leq h_*(a) \vee b$. (h_* is the right adjoint of h.)

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Remark

A frame homomorphism $h : B \to A$ is closed iff the right adjoint h_* maps closed sublocales of A to closed sublocales of B.

Lemma

A frame homomorphism $h : B \to A$ is closed iff for each $a \in A$ there exists $b'' \in B$ such that $h(b'') \leq a$ and, for each $b, b' \in B$, $h(b') \leq a \vee h(b) \Rightarrow b' \leq b'' \vee b$.

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Theorem

For any frame A satisfying UD, and any frame B, the frame injection $q_B : B \to A \oplus B$ is closed.

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Let A satisfy UD, and let B be any frame.

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Let A satisfy UD, and let B be any frame. Take $c \in A \oplus B$ and put $c = \bigvee_{i \in I}(q_A(a_i) \wedge q_B(b_i))$. Put $b'' = \bigvee\{\bigwedge_{i \in K} b_i \mid K \subseteq I, \bigvee_{i \in K} a_i = 1\}$. Then $q_B(b'') \leq c$.

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A frame homomorphism $h : B \to A$ is closed iff for each $a \in A$ there exists $b'' \in B$ such that $h(b'') \leq a$ and, for each $b, b' \in B$, $h(b') \leq a \vee h(b) \Rightarrow b' \leq b'' \vee b$.

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The converse

Proposition

For a frame A, if the injection $q_B : B \to A \oplus B$ is closed for all frames B, then A is compact.

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For a frame A, if the injection $q_B : B \to A \oplus B$ is closed for all frames B, then A is compact.

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Let A be noncompact. Fix a cover U for which for all finite $F \subseteq U$, F does not cover A.

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Proposition

For a frame A, if the injection $q_B : B \to A \oplus B$ is closed for all frames B, then A is compact.

Proof idea.

Let A be noncompact. Fix a cover U for which for all finite $F \subseteq U$, F does not cover A.
Put $A_U = \{M \subseteq A \mid 1 \in M \Rightarrow \exists \text{finite } F \subseteq U \text{ s.t. } \uparrow \bigvee F \subseteq M\}$.

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For a frame A, if the injection $q_B : B \to A \oplus B$ is closed for all frames B, then A is compact.

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For a frame A, if the injection $q_B : B \to A \oplus B$ is closed for all frames B, then A is compact.

Proof idea.

Let A be noncompact. Fix a cover U for which for all finite $F \subseteq U$, F does not cover A. Put $A_U = \{M \subseteq A \mid 1 \in M \Rightarrow \exists \text{finite } F \subseteq U \text{ s.t. } {\uparrow} \bigvee F \subseteq M\}$. Then A_U is a topology on A, so A_U is a frame. Consider the coproduct $A \oplus A_U$. Put $c = \bigvee_{u \in U}(q_A(u) \wedge q_{A_U}({\uparrow} u))$. Let p_{A_U} be the right adjoint of q_{A_U}. Then $p_{A_U}(c) = \emptyset$.

The converse

Proposition

For a frame A, if the injection $q_B : B \to A \oplus B$ is closed for all frames B, then A is compact.

Proof idea.

Let A be noncompact. Fix a cover U for which for all finite $F \subseteq U$, F does not cover A. Put $A_U = \{M \subseteq A \mid 1 \in M \Rightarrow \exists \text{finite } F \subseteq U \text{ s.t. } \uparrow \bigvee F \subseteq M\}$. Then A_U is a topology on A, so A_U is a frame. Consider the coproduct $A \oplus A_U$. Put $c = \bigvee_{u \in U}(q_A(u) \wedge q_{A_U}(\uparrow u))$. Let p_{A_U} be the right adjoint of q_{A_U}. Then $p_{A_U}(c) = \emptyset$. One shows that $1_{A \oplus A_U} = c \vee q_{A_U}(A \setminus \{1\})$.

The converse

Proposition

For a frame A, if the injection $q_B : B \to A \oplus B$ is closed for all frames B, then A is compact.

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The converse

Proposition

For a frame A, if the injection $q_B : B \to A \oplus B$ is closed for all frames B, then A is compact.

Proof idea.

Let A be noncompact. Fix a cover U for which for all finite $F \subseteq U$, F does not cover A. Put $A_U = \{M \subseteq A \mid 1 \in M \Rightarrow \exists \text{finite } F \subseteq U \text{ s.t. } \uparrow \bigvee F \subseteq M\}$. Then A_U is a topology on A, so A_U is a frame. Consider the coproduct $A \oplus A_U$. Put $c = \bigvee_{u \in U}(q_A(u) \wedge q_{A_U}(\uparrow u))$. Let p_{A_U} be the right adjoint of q_{A_U}. Then $p_{A_U}(c) = \emptyset$. One shows that $1_{A \oplus A_U} = c \vee q_{A_U}(A \setminus \{1\})$. But $1_{A \oplus A_U} = q_{A_U}(A)$, so $q_{A_U}(A) \leq c \vee q_{A_U}(A \setminus \{1\})$, while $A \not\subseteq p_{A_U}(c) \cup A \setminus \{1\}$. So q_{A_U} is not closed. $\square$

Theorem

For any frame A, the following are equivalent:

(i) *A is compact.*

(ii) *A satisfies UD.*

(iii) *For any frame B, the injection $q_B : B \to A \oplus B$ is closed.*

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