T_D-coreflection pointfreely and its generalization to other separation axioms

Ranjitha Raviprakash

University of KwaZulu-Natal, South Africa

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Joint work with Guram Bezhanishvili, Sebastian Melzer, and Anna Laura Suarez

Coreflections of lower separation axioms

Let **Top** be the category of topological spaces and continuous maps, and for $i \in \{0, D, 1, 2\}$ let $\mathbf{Top}_i$ be its full subcategory of T_i-spaces.

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The aim of this talk is to obtain analogous Banaschewski–Pultr-style coreflections for T_0, T_1, and T_2, and to describe the construction pointfreely using MT-algebras.

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We will express this coreflector in terms of isolated points.

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The Banaschewski–Pultr coreflector can therefore be understood through the isolated-point construction.

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Proposition (Amini and Golestani, 2016)

$\mathbf{Top}_\Delta$ *is coreflective in* ${}_\Delta\mathbf{Top}$, *with coreflector* $X \mapsto iso(X)$.

Recovering the T_D-coreflection

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Moreover, the morphisms of ${}_D\mathbf{Top}$ are exactly the morphisms needed for the isolated-point coreflector after passing to the Skula topology.

Thus, the construction consists of forming an associated topology and then selecting its isolated points. We now describe this construction pointfreely.

MT-algebras

The lower separation axioms are not treated uniformly in the language of frames. In particular, the T_0 axiom is not visible at the level of frames of opens.

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We therefore use McKinsey–Tarski algebras.[2]

Definition

- An MT-algebra is a complete boolean algebra M together with a subframe $\mathcal{O}(M)$ of open elements.
- An MT-morphism is a complete boolean homomorphism which sends open elements to open elements.
- Let **MT** denote the category of MT-algebras and MT-morphisms.

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Equivalently, an MT-algebra is a complete boolean algebra equipped with an interior operator $\Box$, whose fixed points are the open elements.

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MT-algebras and frames

Taking open elements defines a functor $\mathcal{O}: \mathbf{MT} \to \mathbf{Frm}$. On morphisms, $\mathcal{O}$ restricts an MT-morphism to its open elements.

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By Funayama's theorem, $\mathcal{O}$ is essentially surjective. Every frame L is (isomorphic to) the frame of open elements of an MT-algebra, its Funayama envelope.[3]

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Functoriality can be recovered by replacing MT-morphisms with the weaker proximity morphisms, but we will not need them here.[4]

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Theorem (Bezhanishvili and Raviprakash, 2023)

There is a dual adjunction between **Top** *and* **MT** *which restricts to a dual equivalence between* **Top** *and* **SMT**.

The MT-adjunction

$$
\begin{tikzcd}
\mathbf{Top} \arrow[r, shift left, "\mathscr{P}"] & \mathbf{MT} \arrow[l, shift left, "\mathscr{at}"] \\
\mathbf{Top} \arrow[u, equal] \arrow[r, leftrightarrow, "\cong"] & \mathbf{SMT} \arrow[u, hook]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{Top} \arrow[r, shift left, "\Omega"] & \mathbf{Frm} \arrow[l, shift left, "\mathscr{pt}"] \\
\mathbf{Sob} \arrow[u, hook] \arrow[r, leftrightarrow, "\cong"] & \mathbf{SFrm} \arrow[u, hook]
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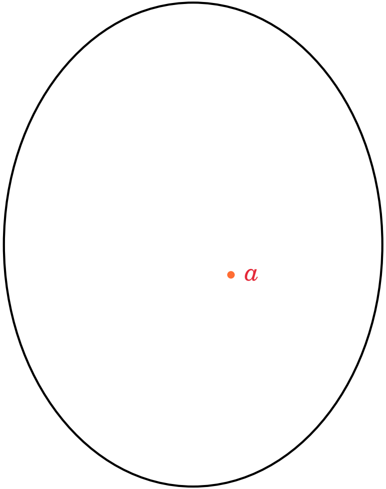

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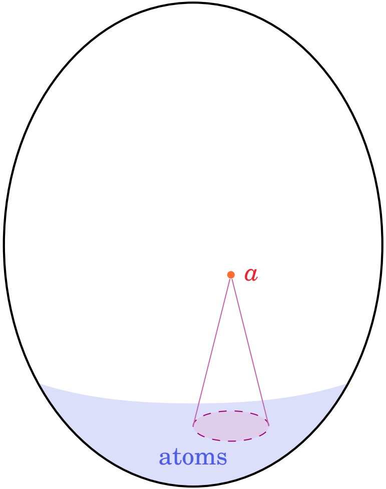

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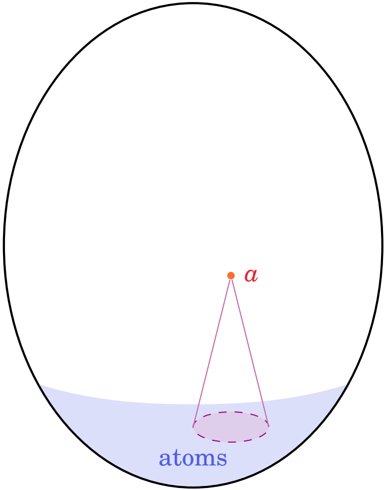
This perspective allows us to describe separation axioms in terms of properties of a join-generating set.



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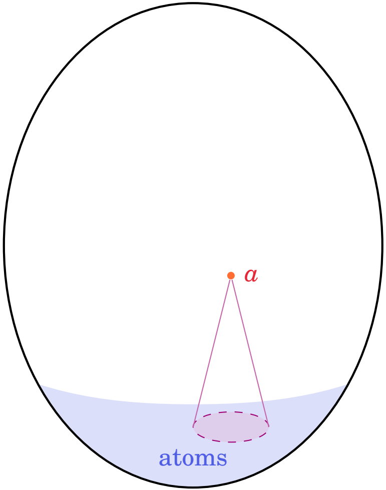

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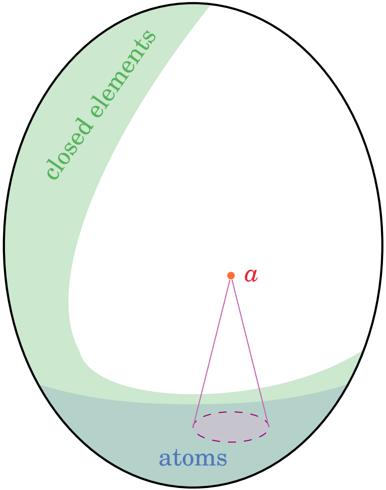

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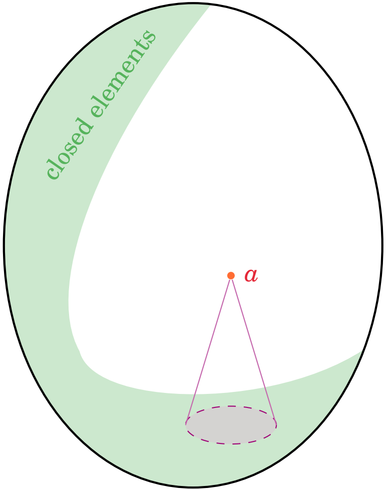

Lower separation axioms in MT-algebras

Let M be an MT-algebra. Write $\mathscr{C}(M) = \{\neg u \mid u \in \mathcal{O}(M)\}$ for its closed elements and $\mathcal{S}(M) = \{\bigwedge_i u_i \mid u_i \in \mathcal{O}(M)\}$ for its saturated elements. Also write $\Diamond u = \neg\Box\neg u$.

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We call $a \in M$

- a T_0-element if $a = s \wedge c$ for some $s \in \mathcal{S}(M)$ and $c \in \mathscr{C}(M)$;
- a T_D-element if $a = u \wedge c$ for some $u \in \mathcal{O}(M)$ and $c \in \mathscr{C}(M)$;
- a T_1-element if $a \in \mathscr{C}(M)$;
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Theorem (Bezhanishvili and Raviprakash, 2023)

$\mathbf{Top}_i$ *is dually equivalent to* $\mathbf{SMT}_i$.

The MT-algebras M_i

For $i \in \{0, D, 1, 2\}$, let $\mathcal{O}_i(M)$ be the subframe of M generated by the T_i-elements, and let $M_i = (M, \mathcal{O}_i(M))$ be the resulting MT-algebra.

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The T_i-elements form a basis for $\mathcal{O}_i(M)$, and so we obtain:

Lemma

The following are equivalent:

1. *M is T_i;*
2. *$\mathcal{O}_i(M) = M$;*
3. *M_i is discrete.*

(An MT-algebra is discrete if all of its elements are open.)

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Thus, each lower separation axiom is equivalent to discreteness of the corresponding MT-algebra M_i.

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Lemma

The element d_M is the largest open discrete element of M.

Example

- If $M = \mathcal{P}(X)$, then $d_M = iso(X)$.
- If M is discrete, then $d_M = 1$ (even if M is atomless).

The pointfree isolated-point reflection

As for spaces, this construction is not functorial for arbitrary MT-morphisms. We therefore restrict to those morphisms $h : M \to N$ satisfying $d_N \leq h(d_M)$, which ensures that h restricts to the corresponding relativizations.

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As for spaces, this construction is not functorial for arbitrary MT-morphisms. We therefore restrict to those morphisms $h : M \to N$ satisfying $d_N \leq h(d_M)$, which ensures that h restricts to the corresponding relativizations.

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For a topological space X,

$$d_{\mathscr{P}(X)} = iso(X) \qquad \text{and} \qquad \mathscr{P}(X)_{d_{\mathscr{P}(X)}} \cong \mathscr{P}(iso(X)).$$

Thus, spatially this reflection recovers the isolated-point coreflection.

T_i-reflections of MT-algebras

Recall that M is T_i precisely when $M_i = (M, \mathcal{O}_i(M))$ is discrete. We may therefore apply the preceding construction to M_i.

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If M and N are T_i, then $d_i(M) = d_i(N) = 1$, so every MT-morphism $M \to N$ is automatically a T_i-morphism. Hence $\mathbf{MT}_i$ is a full subcategory of ${}_i\mathbf{MT}$.

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Theorem

For each $i \in \{0, D, 1, 2\}$, $\mathbf{MT}_i$ is reflective in ${}_i\mathbf{MT}$. The reflector sends M to $M_{d_i(M)}$.

The case of T_D

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Theorem

The Skula-discrete relativization of an MT-algebra is its T_D-reflection.

For spatial MT-algebras, this reflection dualizes to the Banaschewski–Pultr T_D-coreflection.

T_i-coreflections of spaces

Let X be a topological space. The T_i-elements of $\mathscr{P}(X)$ generate a topology on the underlying set of X.

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A point is T_i-isolated if it is isolated in this topology. Write $iso_i(X)$ for the set of all T_i-isolated points. Concretely, a point $x \in X$ is

- T_0-isolated if $\{x\}$ is the intersection of a saturated set and a closed set;
- T_D-isolated if $\{x\}$ is locally closed;
- T_1-isolated if $\{x\}$ is closed;
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A continuous map is a T_i-map if it preserves T_i-isolated points. Let ${}_i\mathbf{Top}$ be the category of spaces and T_i-maps.

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Corollary

For each $i \in \{0, D, 1, 2\}$, $\mathbf{Top}_i$ is coreflective in ${}_i\mathbf{Top}$, with coreflector $X \mapsto \mathit{iso}_i(X)$.

Conclusion

The Banaschewski–Pultr T_D-coreflection is obtained by passing to the Skula topology and then selecting its isolated points.

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For each $i \in \{0, D, 1, 2\}$, the T_i-reflection is obtained by first forming M_i and then applying the same pointfree isolated-point construction.

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Restricting to spatial MT-algebras yields Banaschewski–Pultr-style coreflections for all four lower separation axioms.

Further directions

There is also a frame-theoretic route for T_D, T_1, and T_2.

[5] Y. Maruyama. “Topological duality via maximal spectrum functor”. In: *Comm. Algebra* 48.6 (2020), pp. 2616–2623.

Further directions

There is also a frame-theoretic route for T_D, T_1, and T_2.

The T_D-spatial reflection is due to Banaschewski–Pultr, the T_1-spatial reflection follows from Maruyama's duality,[5] and an analogous T_2-spatial reflection yields a duality with Hausdorff spaces.

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These constructions yield coreflections of spaces but, on the frame side, only reflections into the corresponding spatial frames.

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We are also developing a sublocale-based construction of T_i-reflections for arbitrary frames, parallel to the MT-algebraic construction discussed here, and comparing the two constructions. They need not agree in general.

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Thank you

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