Easy Direct Limits Property in Some Classes of Algebras

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Outline

Introduction

What can be obtained from one algebra by direct limits?

Introduction

Background

The direct limits construction is basic in Universal algebra and in Category theory. It builds up new algebras from directed systems of algebras (homomorphisms are included).

- A natural question is: if we start with one algebra A, how complicated can the algebras obtained from A by direct limits be?
- The talk focuses on the most lean case: every such direct limit is already represented by a retract of A.

Motivation

1. a general study of the construction
2. a study of direct limit closed classes

Motivation 1: General properties of the construction

Grätzer G. Universal algebra, 2nd edition with updates, Springer Science+Business Media, LLC, 2008.

Chapter 3 - Constructions of algebras, p. 128:

§21. DIRECT AND INVERSE LIMITS OF ALGEBRAS

There are two well known methods to build up algebras from families of algebras, the so-called direct and inverse limits. No systematic account of the properties of these constructions can be found in the literature, although almost all the results given below belong to the "folklore". Besides, most results are set theoretic in nature, and they can be derived from the case of groups; see for example S. Eilenberg and W. Steenrod, Foundations of algebraic topology, Princeton University Press, Princeton, N. J., 1952.†

Motivation 1: General properties of the construction

- a new one will be presented in Section 4
- derived from mono-unary algebras are in the papers:
  - E. Halušková, M. Ploščica:On direct limits of finite algebras, Contributions to general algebra, 11 (Olomouc, 1998,Velke Karlovice, 1998), 101–104,Verlag Johannes Heyn, Klagenfurt, 1999.
  - E. Halušková: Algebras with one and two direct limits, Contributions to general algebra, 12 (Vienna, 1999), 211–219, Heyn, Klagenfurt,2000.

Grätzer G. Universal algebra, 2nd edition with updates, Springer Science+Business Media, LLC, 2008.

Problem on the page 161:

26. For an algebraic class K, define K^α for every ordinal α as follows: $K^0 = K$, $K^{\alpha+1} = \underrightarrow{\mathbf{IL}}(K^\alpha)$, $K^{\lim \alpha_\nu} = \bigcup (K^{\alpha_\nu} \mid \nu)$. Is it possible to find for every ordinal α a class such that $K^{\alpha+1} \neq K^\alpha$? Is it true that for every class K there exists an α with $K^\alpha = K^{\alpha+1}$? If this is so, which ordinals can occur as a smallest such α? (See Ex. 82.)

Problem on the page 161:

26. For an algebraic class K, define K^α for every ordinal α as follows: $K^0 = K$, $K^{\alpha+1} = \mathbf{I}\underset{\rightarrow}{\mathbf{L}}(K^\alpha)$, $K^{\lim \alpha_\nu} = \bigcup (K^{\alpha_\nu} \mid \nu)$. Is it possible to find for every ordinal α a class such that $K^{\alpha+1} \neq K^\alpha$? Is it true that for every class K there exists an α with $K^\alpha = K^{\alpha+1}$? If this is so, which ordinals can occur as a smallest such α? (See Ex. 82.)

- a class for $\alpha = 1$ is in the paper
  E. Halušková: On iterated direct limits of a monounary algebra, Contributions to general algebra, 10 (Klagenfurt, 1997), Austria, 189–195, Verlag Johannes Heyn, Klagenfurt, 1998.
- nothing new will be presented in this talk

Motivation 2: Direct limit closed classes

Theory of Models with Generalized Atomic Formulas
Author(s): H. Jerome Keisler
Source: *The Journal of Symbolic Logic*, Vol. 25, No. 1 (Mar., 1960), pp. 1-26

THEOREM 6. *If* $\mathbf{K} \in \mathrm{EC}_\delta$, *then* $[L]$-Sub(F-DirLim($\mathbf{K}$)) $= \exists \vee \wedge F$-Sub($\mathbf{K}$) $= (\mathbf{K}^* \cap \forall \mathbf{B} \exists \vee \wedge F)^*$; *that is,* $\mathfrak{A}$ *has an elementary extension isomorphic to an F-direct limit of an F-direct system of elements of* $\mathbf{K}$ *if and only if* $\mathfrak{A}$ *is an* $\exists \vee \wedge F$*-subsystem of an element of* $\mathbf{K}$, *and this is true if and only if every sentence of* $\forall \mathbf{B} \exists \vee \wedge F$ *holding throughout* $\mathbf{K}$ *also holds in* $\mathfrak{A}$.

In this connection we observe that the sentences of $\forall \mathbf{B} \exists \vee \wedge F$ are precisely those sentences equivalent to a sentence of the form

$$\forall x_1 \ldots x_s \left[\bigwedge_{j=1}^{r} (g_j(x_1, \ldots, x_s) \supset \exists x_{s+1} \ldots x_t \, h_j(x_1, \ldots, x_t)) \right]$$

where $g_j, h_j \in \vee \wedge F$, $(j = 1, \ldots, r)$.

Motivation 2: Direct limit closed classes

E. Halušková: On direct limit closed classes of algebras, Bulletin of the Section of Logic 45:3/4(2016), 155-169.

Let Γ be the class of all sentences γ of $F_{\infty\infty}$ such that γ is equivalent to a sentence of the form

$$(\forall\ x_0, \dots, x_{n-1})(\ \alpha \to (\exists\ y_\mu;\ \ \mu < \varrho)\ \beta\ ),$$

where α is a formula of $F_{\omega 0}$, β is a positive formula of $F_{\infty 0}$, $n \in \mathbb{N}_0$ and $\varrho \in Ord$.

Let Δ be the class of all sentences γ of $F_{\omega\omega}$ such that γ is equivalent to a sentence of the form

$$\bigwedge_{i=1}^{m} (\forall\ x_1, \dots, x_s)\ (\alpha_i \to (\exists\ x_{s+1}, \dots, x_t)\beta_i),$$

where $m \in \mathbb{N}$, α_i, β_i are positive formulas of $F_{\omega 0}$ for every $i = 1, \dots, m$, $s, t \in \mathbb{N}_0$, $s \leq t$.

Motivation 2: Direct limit closed classes

E. Halušková: On direct limit closed classes of algebras, Bulletin of the Section of Logic 45:3/4(2016), 155-169.

Theorem 3 says that every axiomatic class determined by some first-order sentences is closed with respect to direct limits if and only if it is an axiomatic class determined by some sentences from Δ. The question, whether an analogous statement about axiomatic classes generally and the class of sentences Γ is valid, is open. It is interesting even in the case of elementary classes.

II

Preliminaries

Universal algebras, retracts, direct limits and term operations

Universal algebra

Definition

An algebra is a pair

$$\mathbf{A} = (A, F),$$

where A is a non-empty underlying set and F is a set of finitary operations on A.

The set F need not be finite and may even be empty.

- Examples in the talk include groups, monoids, unary algebras, and mono-unary algebras.

Homomorphisms and endomorphisms

Let $\mathcal{A} = (A, F)$ and $\mathcal{B} = (B, F)$ be algebras of the same type.

Homomorphism

A mapping $\varphi : A \to B$ is a homomorphism if, for each n-ary $g \in F$,

$$\varphi(g(a_1, \ldots, a_n)) = g(\varphi(a_1), \ldots, \varphi(a_n)).$$

An endomorphism is a homomorphism from $\mathcal{A}$ to itself.

Retracts

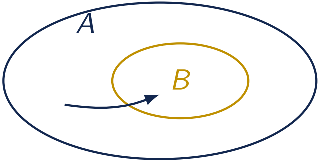
Suppose that B is a subalgebra of A.

Definition

B is a retract of A if there exists an endomorphism φ of A such that

$$\varphi(A) = B \quad \text{and} \quad \varphi(b) = b \quad \text{for every } b \in B.$$

The map φ is called a retraction of A onto B.

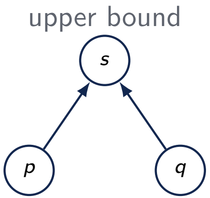


$\varphi|_B = \mathrm{id}_B$

Directed partially ordered sets

Definition

A partially ordered set $(P, \leq)$ is directed if every finite subset has an upper bound.



Direct systems

Let P be a directed partially ordered set. For each $p \in P$ there is an algebra (A_p, F) of the same fixed type.

For $p < q$ there is a homomorphism

$$\varphi_{pq} : A_p \to A_q.$$

The homomorphisms are closed under composition:

$$\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs} \quad \text{for } p < q < s.$$

Also φ_{pp} is the identity on A_p.

$$\begin{tikzcd}
A_p \arrow[r, "\varphi_{pq}"] \arrow[rr, bend right, "\varphi_{ps}"'] & A_q \arrow[r, "\varphi_{qs}"] & A_s
\end{tikzcd}$$

Direct limit: equivalence relation

Assume the sets A_p are pairwise disjoint. For $x \in A_p$ and $y \in A_q$, define

$$x \equiv y$$

if there exists $s \in P$ with $p \leq s$ and $q \leq s$ such that

$$\varphi_{ps}(x) = \varphi_{qs}(y).$$

Direct limit: notation and operations

For each $z \in \bigcup_{p\in P} A_p$ put $\overline{z} = \{t \in \bigcup_{p\in P} A_p : z \equiv t\}$.

In the set $\{\overline{z} : z \in \bigcup_{p\in P} A_p\}$ we define operations:

If $g \in F$ is n-ary, choose representatives $x_j \in A_{p_j}$ and an upper bound s of all p_j. Then operations are defined by transporting representatives to A_s:

$$g(\bar{x}_1, \dots, \bar{x}_n) = \overline{g(\varphi_{p_1 s}(x_1), \dots, \varphi_{p_n s}(x_n))}.$$

The set $\overline{A} = \{\overline{z} : z \in \bigcup_{p\in P} A_p\}$ with operations defined above is an algebra known as the direct limit algebra and is denoted by $\bar{\mathcal{A}}$.

$$\{P, A_p, \varphi_{pq}\} \longrightarrow \bar{\mathcal{A}} = (\bar{A}, F). \tag{2.1}$$

A-uniform direct families and $\varinjlim A$

A direct family is called *A-uniform* if every A_p is isomorphic to the same algebra A.

Definition of $\varinjlim A$, R_A

$$\begin{aligned}\varinjlim A &= \text{the class of all isomorphic copies of direct limits obtainable from } A,\\ R_A &= \text{the set of all retracts of } A.\end{aligned}$$

Every retract of A can be obtained, up to isomorphism, as a direct limit of A. Hence

$$R_A \subseteq \varinjlim A.$$

Term operations $T(F)$

Let F be a set of fundamental operations of an algebra.

The set $T(F)$ of term operations over F is the smallest set satisfying the following conditions:

- it contains each basic operation $f \in F$;
- it contains all coordinate projections

$$p_i^n(a_1, \ldots, a_n) = a_i;$$

- it is closed under composition.

III

Mono-unary algebras

Directed graphs with exactly one outgoing arrow from each vertex

Definition: mono-unary algebra

Definition

A mono-unary algebra is a pair

$$(A, h),$$

where $h : A \to A$ is a unary operation.

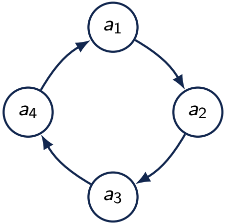
It can be represented as a directed graph:

- each element $a \in A$ is a vertex;
- for each $a \in A$, draw one directed edge $a \to h(a)$;
- every vertex has exactly one outgoing arrow.

Cycles

A cycle of length k consists of elements $a_1, \ldots, a_k$ such that

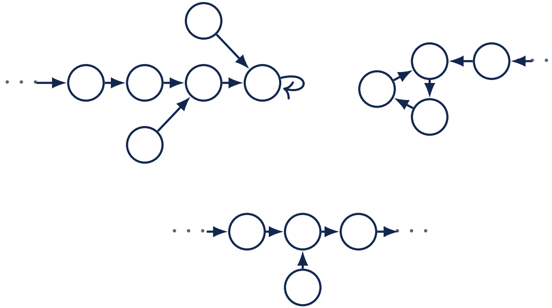
$$h(a_i) = a_{i+1} \quad (1 \leq i < k), \qquad h(a_k) = a_1.$$



A fixed point of h is a cycle of length 1.

A connected mono-unary algebra may contain a cycle; if it does, the cycle is unique inside that component.

Figure 1: a mono-unary algebra



The figure visualizesa mono-unary algebra which consists of 3 connected components.

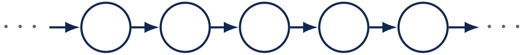
Figure 2: the algebra $\mathbb{Z}$, a line

The mono-unary algebra $\mathbb{Z}$ is defined on the set of all integers with the successor operation.

$$\cdots \longrightarrow -2 \to -1 \to 0 \to 1 \to 2 \longrightarrow \cdots$$

Definition

A line is a mono-unary algebra isomorphic to $\mathbb{Z}$ with the successor operation.



Characterization of lines

Lemma 1

Let (A, h) be a mono-unary algebra. Then the following statements are equivalent:

1. (A, h) is a line.
2. (A, h) is connected, A is infinite, and the operation h is bijective.
3. (A, h) is connected without a cycle and the operation h is bijective.

Disjoint sums of components

If I is a nonempty set and (B_i, h) are mono-unary algebras, their disjoint union is denoted by

$$\sum_{i \in I} (B_i, h).$$

Every mono-unary algebra can be decomposed as a disjoint union of connected components:

$$(A, h) = \sum_{i \in I} (B_i, h).$$

Diamond construction: definition

Let

$$(A, h) = \sum_{i \in I} (B_i, h),$$

where every (B_i, h) is connected. For each $i \in I$:

- if (B_i, h) contains a cycle of length k, let (C_i, h) be a cycle of length k;
- otherwise, let (C_i, h) be a line.

The diamond algebra is

$$(A, h)^\diamond = \sum_{i \in I} (C_i, h).$$

(The diamond construction is applied component by component.)

IV

Main construction and main theorem

Lemma 3

Let φ be a homomorphism from an algebra (A, F) into (A', F) and let $g \in T(F)$. Then φ is a homomorphism from the algebra (A, g) into (A', g).

Meaning:

$$\varphi(g(a_1, \ldots, a_n)) = g(\varphi(a_1), \ldots, \varphi(a_n)).$$

Compatibility with the fundamental operations automatically implies compatibility with every term operation.

Lemma 4

Let $D = (D, F)$ be an algebra and $g \in T(F)$. Suppose that $\{P, A_p, \varphi_{pq}\}$ is a direct family and

$$\{P, A_p, \varphi_{pq}\} \longrightarrow \bar{A} = (\bar{A}, F).$$

If $\bar{A}$ is isomorphic to a retract of D, then $(\bar{A}, g)$ is isomorphic to a retract of (D, g).

This lemma allows us to transfer retraction information from the original algebra to the mono-unary reduct determined by a term.

Lemma 5

Let $A = (A, F)$ and let f be an endomorphism of A. Define, for each $i \in \mathbb{N}$,

$$A_i = \{(a, i) : a \in A\}.$$

The operations on A_i are copied from A: $g((a_1, i), \ldots, (a_n, i)) = (g(a_1, \ldots, a_n), i)$.
For $i \leq j$, define

$$\varphi_{ij}(a, i) = (f^{j-i}(a), j).$$

Then $\{\mathbb{N}, A_i, \varphi_{ij}\}$ is an A-uniform direct system.

Let

$$\{\mathbb{N}, A_i, \varphi_{ij}\} \longrightarrow \bar{A} = (\bar{A}, F).$$

This direct limit is built only from copies of A, so

$$\bar{A} \in \underrightarrow{\mathbf{L}} A.$$

If A has easy direct limits, then $\bar{A}$ is isomorphic to a retract of A.

Main Theorem

Let $A = (A, F)$. Suppose that $\{\mathbb{N}, A_i, \varphi_{ij}\}$ is the A-uniform direct system from Lemma of the previous slide and $\{\mathbb{N}, A_i, \varphi_{ij}\} \longrightarrow \bar{A} = (\bar{A}, F)$.

Lemma 6

If $F = \{h\}$, where h is unary then

$$(\bar{A}, h) \cong (A, h)^\diamond.$$

Main Theorem

Let $f \in T(F)$ be a unary term such that f is an endomorphism of (A, F). Then

$$(\bar{A}, f) \cong (A, f)^\diamond.$$

If (A, F) is an algebra with easy direct limits, then the mono-unary algebra $(A, f)^\diamond$ is isomorphic to a retract of (A, f).

Corollary

Corollary

Let $A = (A, F)$ and let $f \in T(F)$ be a unary term such that f is an endomorphism of the algebra A. Then:

1. there exists $B = (B, F) \in \underrightarrow{\mathbf{L}}\, A$ such that f is bijective on B;
2. if A is an algebra with easy direct limits, then there exists $B = (B, F) \in R_A$ such that f is bijective on B.

How to use the theorem

The theorem provides a practical easy direct limit test for algebras which have a unary term that is an endomorphism.

1. Choose f which is a unary term over F and an endomorphism of (A, F).
2. Draw or analyze the mono-unary algebra (A, f).
3. Construct $(A, f)^\diamond$.
4. Check whether $(A, f)^\diamond$ can be a retract of (A, f).
5. If not, then (A, F) is not an algebra with easy direct limits.

$$(A, f)^\diamond \ncong \text{ a retract of } (A, f) \quad \Longrightarrow \quad (A, F) \text{ is not with easy direct limits.}$$

V

Applications

Integers, reals, and rationals through mono-unary graphs

Example 1: additive group of integers

Consider the additive group

$$(\mathbb{Z}, +, -, 0).$$

Let

$$f(x) = x + x = 2x.$$

The operation f is a term operation of the group language. Since the group is abelian, f is an endomorphism:

$$f(x + y) = 2(x + y) = 2x + 2y = f(x) + f(y).$$

Now study the mono-unary algebra $(\mathbb{Z}, f)$.

Figure 4: infinite components of $(\mathbb{Z}, f)$

For $f(x) = 2x$, every nonzero infinite component is generated by an odd integer $2k+1$:

$$2k+1 \longrightarrow 2(2k+1) \longrightarrow 4(2k+1) \longrightarrow 8(2k+1) \longrightarrow \cdots$$

There is also the one-element component $\{0\}$.

Observation

No subalgebra of $(\mathbb{Z}, f)$ contains a line, because the backward chain needed for a line is absent.

Why $(\mathbb{Z}, +, -, 0)$ is not an algebra with easy direct limits

The mono-unary algebra $(\mathbb{Z}, f)$ has one component $\{0\}$ and infinitely many infinite components generated by odd numbers.

- The diamond algebra $(\mathbb{Z}, f)^\diamond$ consists of one one-element cycle and infinitely many lines.
- But $(\mathbb{Z}, f)$ contains no line as a subalgebra.
- Therefore $(\mathbb{Z}, f)^\diamond$ is not isomorphic to a retract of $(\mathbb{Z}, f)$.

By Main Theorem,

$$(\mathbb{Z}, +, -, 0) \quad \text{is not an algebra with easy direct limits.}$$

Example 2: multiplicative real monoid

Consider the multiplicative monoid
$$(\mathbb{R}, \cdot, 1).$$
Let
$$g(x) = x \cdot x = x^2.$$
The operation g is a term operation. Commutativity implies that g is an endomorphism:
$$g(xy) = (xy)^2 = x^2y^2 = g(x)g(y).$$
The mono-unary algebra $(\mathbb{R}, g)$ has uncountably many components.

Figure 5: the component of $(\mathbb{R}, g)$ containing 3

In the real case, square roots create backward branches. Every infinite component contains a line.

$$
\begin{tikzcd}
\cdots \arrow[r] & \sqrt{3} \arrow[r] & 3 \arrow[r] & 3^2 \arrow[r] & 3^4 \arrow[r] & 3^8 \arrow[r] & \cdots \\
& -\sqrt{3} \arrow[ur] & -3 \arrow[ur] & -3^2 \arrow[ur] & -3^4 \arrow[ur] &
\end{tikzcd}
$$

The presence of real square roots distinguishes the real case from the rational case.

Why the theorem is ineffective for $\mathbb{R}$

Finite components include

$$\{0\} \quad \text{and} \quad \{-1, 1\}.$$

Other components are infinite and mutually isomorphic. Every infinite component contains a line as a subalgebra.

- By Lemma 2, the relevant cycles and lines can be retracts.
- Thus $(\mathbb{R}, g)^\diamond$ is isomorphic to a retract of $(\mathbb{R}, g)$.

Therefore Main Theorem does not decide whether

$$(\mathbb{R}, \cdot, 1)$$

has easy direct limits.

Example 3: multiplicative rational monoid

Now consider the monoid

$$(\mathbb{Q}, \cdot, 1).$$

Again use

$$g(x) = x^2.$$

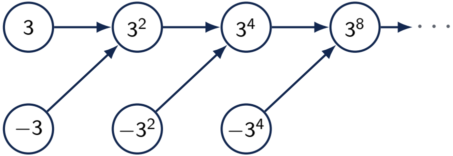
The term is an endomorphism for the same commutativity reason:

$$g(xy) = x^2y^2.$$

However, the mono-unary structure of $(\mathbb{Q}, g)$ is different because many square roots are not rational.

Figure 6: the component of $(\mathbb{Q}, g)$ containing 3

In the rational case, missing square roots prevent the existence of a line as a subalgebra.



This missing backward chain is exactly the obstruction detected by Main Theorem.

Why $(\mathbb{Q}, \cdot, 1)$ is not an algebra with easy direct limits

The mono-unary algebra $(\mathbb{Q}, g)$ has infinitely many components, and the infinite components are isomorphic to each other.

- There is no line as a subalgebra of $(\mathbb{Q}, g)$.
- The diamond algebra $(\mathbb{Q}, g)^\diamond$ contains lines.
- Therefore $(\mathbb{Q}, g)^\diamond$ is not isomorphic to a retract of $(\mathbb{Q}, g)$.

By Main Theorem,

$$(\mathbb{Q}, \cdot, 1)$$

is not an algebra with easy direct limits.

Comparison of the three main examples

Structure	Term	Mono-unary feature	Diamond/retract test	Conclusion
$(\mathbb{Z}, +, -, 0)$	$f(x) = 2x$	no line in $(\mathbb{Z}, f)$	$(\mathbb{Z}, f)^\diamond$ has lines, not a retract	not easy
$(\mathbb{R}, \cdot, 1)$	$g(x) = x^2$	infinite components contain lines	diamond algebra is a retract	inconclusive
$(\mathbb{Q}, \cdot, 1)$	$g(x) = x^2$	no line in $(\mathbb{Q}, g)$	$(\mathbb{Q}, g)^\diamond$ has lines, not a retract	not easy

References

- E. Halušková and M. Jastrzębska, *Mono-unary condition for algebras with easy direct limits*, Altay Conference Proceedings in Mathematics, 1(1), 2025, 1–9.
- G. Grätzer, *Universal Algebra*, Springer, 2008 edition with updates.
- E. Halušková, *Monounary algebras with easy direct limits*, Miskolc Mathematical Notes, 19(1), 2018, 291–302.

Additional references are listed in the first article.

Thank you