Galois Theory of Differential Schemes

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CLASSICAL PICARD-VESSIOT THEORY

$$\begin{tikzcd} (A, \delta_A) \arrow[rr, hook] & & (L, \delta_L) \\ & (K, \delta_K) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

PV extension of differential fields with

- PV ring (A, δ_A)
- $k = \mathrm{Const}(K) = \mathrm{Const}(L)$.

$\rightsquigarrow$ PV Galois group $G = \mathrm{Gal}^{PV}(L/K) = \mathrm{Spec}(\mathrm{Const}(A \otimes_K A))$

linear alg. group / k

$$\left\{ \begin{matrix} \text{intermediate differential} \\ \text{field extensions} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{closed subgroups} \\ \text{of } G \end{matrix} \right\}$$

JANELIDZE'S CATEGORICAL FORMULATION

CATEGORICAL GALOIS THEORY :

$$\begin{tikzcd} \delta\text{-}\mathrm{Ring}^{op} \arrow[d, bend right, "\mathrm{Const}"'] \arrow[d, phantom, "\dashv"] & \simeq & \delta\text{-}\mathrm{Aff} \\ \mathrm{Ring}^{op} \arrow[u, bend right, "(-,0)"'] & \simeq & \mathrm{Aff} \end{tikzcd}$$

$$\begin{tikzcd} X = \mathrm{Spec}(A) \arrow[r, "f"] \arrow[dr] & Y = \mathrm{Spec}(K) \arrow[d] \\ & S = \mathrm{Spec}(k) \end{tikzcd}$$

is GALOIS, NOT L/K.

Categorical Galois group(oid)

$$G = \mathrm{Gal}[f] \simeq \mathrm{Gal}^{PV}(L/K).$$

Self-splitting :

$$X \times_Y X \simeq X \times_{(S,0)} (G,0)$$

Torsor relation

Equivalence :

$$\mathrm{Split}[f] \simeq \mathrm{Aff}^G$$

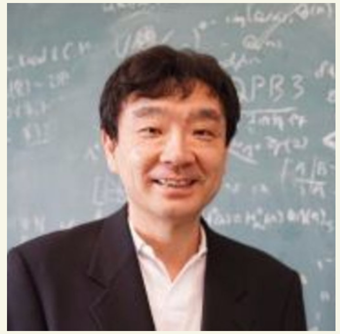
objects in $\delta\text{-}\mathrm{Aff}/_Y$ split by f

G-actions in Aff.

CARBONI-JANELIDZE-MAGID CORRESPONDENCE

$$\left\{ \text{SPLIT QUOTIENTS of } \begin{matrix} X \\ \downarrow \\ Y \end{matrix} \right\} \overset{1:1}{\rightsquigarrow} \left\{ \text{EFFECTIVE SUBGROUPS of } G \right\}$$

$$X/H \longleftarrow\!\shortmid\; H \leq G$$



MASUOKA (Sept 2023): Does Categorical Galois theory recover classical PV correspondence?

Answer: NO! $H \leq G$ is effective in Aff iff G/H affine scheme.

In general G/H can be QUASI-PROJECTIVE.

$\rightsquigarrow$ We must work with GENERAL DIFFERENTIAL SCHEMES.

$$\begin{tikzcd} \delta\text{-Sch} \arrow[d, dashed, bend right=50, "?"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{Sch} \arrow[u, bend right=50, "C=(-,0)"'] \end{tikzcd}$$

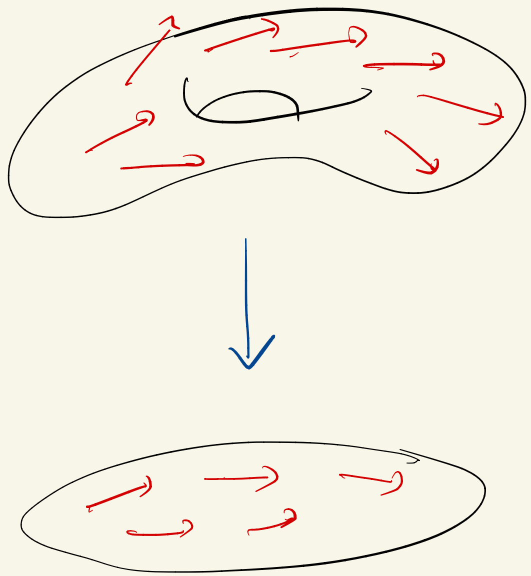
Classical Picard-Vessiot

1883 - 2023

- $$\begin{array}{c}(A,\delta)\\ \big| \\ (K,\delta)\end{array}$$ PV ring extension
- $k = \mathrm{Const}(A,\delta)$
- (A,δ) simple

Noohi - T.

2024 —



- $$\begin{array}{c}(X,\delta)\\ \downarrow \\ (Y,\delta)\end{array}$$ PV morphism
- $\pi_0(X)$ categorical scheme of leaves
- (X,δ) categorically simple

DIFFERENTIAL SCHEMES

S - scheme

S - differential scheme : $(X, (\mathcal{O}_X, \delta_X))$, where $(X, \mathcal{O}_X) \in Sch/_S$

$\delta_X \in Der_S(\mathcal{O}_X, \mathcal{O}_X)$.

a vector field on X.

$\rightsquigarrow$ category

$$\delta\text{-}Sch_S$$

Functor

$$C : Sch/_S \longrightarrow \delta\text{-}Sch_S$$

$$(X, \mathcal{O}_X) \longmapsto (X, (\mathcal{O}_X, 0)).$$

KEY INNOVATIONS

DIFFERENTIAL SCHEMES

(X, δ_X)

PRECATEGORY ACTIONS

$$X_2 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; X_1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X_0$$

- DESCENT
- CATEGORICAL SIMPLICITY
- CATEGORICAL GALOIS THEORY

FORMAL GROUP ACTIONS

$$\hat{G}_a \times X \longrightarrow X$$

- GEOMETRIC QUOTIENTS
- EFFECTIVE CRITERIA for $\pi_0(X)$ and SIMPLICITY
- APPLICATIONS

PRECATEGORIES

$\mathcal{A}$ category,

$\mathbb{C} \in \mathrm{PreCat}(\mathcal{A})$:

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "r_0"] \arrow[r, "m"] \arrow[r, shift right=2, "r_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "n"'] \end{tikzcd}$$

commuting triangles

morphisms

objects

"partial multivalued composition"

$\mathbb{C} \in \mathrm{Cat}(\mathcal{A})$ if

$$C_2 \simeq C_1 \times_{C_0} C_1$$

PRECATEGORY ACTIONS

$\mathcal{P} : \mathcal{A}^{op} \longrightarrow \mathrm{CAT}$ pseudofunctor / indexed cat.

$\mathbb{C}$ - actions in $\mathcal{P}$: $\mathcal{P}^{\mathbb{C}}$

AS DESCENT DATA

(P_0, φ) :

- $P_0 \in \mathcal{P}(C_0)$
- $d_1^* P_0 \xrightarrow{\varphi} d_0^* P_1$ in $\mathcal{P}(C_1)$
- cocycle cond. in $\mathcal{P}(C_2)$

FIBRATIONAL VIEW

$$\begin{tikzcd} \int \mathcal{P} \arrow[d] \\ \mathcal{A} \end{tikzcd}$$

all maps cartesian

$\mathbb{P} \in \mathrm{PreCat}(\int \mathcal{P})$

$\mathbb{C}$

$$\begin{tikzcd} P_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, dashed] & P_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, dashed] & P_0 \arrow[l] \arrow[d, dashed] \\ C_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & C_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & C_0 \arrow[l] \end{tikzcd}$$

DIFFERENTIAL SCHEMES AS PRECATEGORY ACTIONS

PRECATEGORY in $Sch_{/S}$

$$D(S) : \quad S[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_1\varepsilon_2, \varepsilon_2^2) \rightrightarrows\!\!\!\rightarrow S[\varepsilon]/(\varepsilon^2) \;\underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}}\; S$$

Key observation :

$$\delta\text{-}Sch_S \simeq \left(Sch_{/S}\right)^{D(S)}$$

$$(X, \delta_X) \leftrightsquigarrow \mathbb{X} = \left(X_2 \rightrightarrows\!\!\!\rightarrow X_1 \;\underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}}\; X_0 \right)$$

CATEGORICAL SCHEME of LEAVES :

$$\pi_0(X, \delta_X) := \pi_0(\mathbb{X}) = \mathrm{coeq}\left(X_1 \rightrightarrows X_0\right)$$

when it exists in Sch.

$\rightsquigarrow$ partial adjoint

$$\begin{tikzcd} \delta\text{-}Sch_S \arrow[d, dashed, bend right, "\pi_0"'] \arrow[d, phantom, "\dashv"] \\ Sch \arrow[u, bend right, "c"'] \end{tikzcd}$$

DIFFERENTIAL DESCENT

$$\mathcal{P} : (\mathrm{Sch}/_S)^{op} \longrightarrow \mathrm{CAT} \quad \rightsquigarrow \quad \delta\text{-}\mathcal{P} : (\delta\text{-}\mathrm{Sch}_S)^{op} \longrightarrow \mathrm{CAT}$$

$$(X, \delta_X) \longmapsto \mathcal{P}^{\mathbb{X}}$$

precategory $\mathbb{X}$ actions in $\mathcal{P}$.

$$\begin{tikzcd} (X, \delta_X) \arrow[d, "f"'] \\ (Y, \delta_Y) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathbb{X} \arrow[d] & \cdots & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "f_2"'] & X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "f_1"] & X_0 \arrow[l] \arrow[d, "f_0"] \\ \mathbb{Y} & \cdots & Y_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Y_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & Y_0 \arrow[l] \end{tikzcd}$$

__Th__ f is effective descent for δ-$\mathcal{P}$ if f_0 effective descent for $\mathcal{P}$
& f_1 descent for $\mathcal{P}$.

[& f_2 weak descent for $\mathcal{P}$]

CATEGORICAL SIMPLICITY

<u>Def</u> (X, δ_X) is SIMPLE wrt $\mathcal{P} : (\mathrm{Sch}_{/S})^{op} \to \mathrm{CAT}$

if $\pi_0(X, \delta_X) = \pi_0(\mathbb{X})$ exists and is universal for $\mathcal{P}$.

$\rightsquigarrow$ $\eta_X :$

$$\begin{tikzcd} \mathbb{X} \arrow[d] & \cdots & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & X_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X_0 \arrow[l] \arrow[d] \\ \Delta(\pi_0(X)) & \cdots & \pi_0(X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \pi_0(X) \arrow[r, shift left] \arrow[r, shift right] & \pi_0(X) \arrow[l] \end{tikzcd}$$

is of PRECATEGORICAL DESCENT,

i.e., constant object functor

$$\begin{tikzcd} \mathcal{P}(\pi_0(X)) \arrow[r, "c_X"] \arrow[d, equal] & \delta\text{-}\mathcal{P}(X, \delta_X) \arrow[d, equal] \\ \mathcal{P}^{\Delta(\pi_0(\mathbb{X}))} \arrow[r, "\eta_X^*"] & \mathcal{P}^{\mathbb{X}} \end{tikzcd}$$

is fully faithful.

DIFFERENTIAL SCHEMES AS FORMAL GROUP ACTIONS

$$\delta\text{-Sch}_{\mathbb{Q}} \simeq \mathrm{Sch}_{\mathbb{Q}}^{\hat{\mathbb{G}}_a}$$

$$(X, \delta_X) \rightsquigarrow \hat{\mathbb{G}}_a \times X \longrightarrow X$$

$$\mathcal{O}_X(U)[[t]] \longleftarrow \mathcal{O}_X(U)$$

$$\sum_i \frac{\delta_X^i f}{i!} t^i \longleftarrow\!\shortmid f$$

Define : $(X, \delta_X) \to (Q, 0)$ GEOMETRIC QUOTIENT [Mumford GIT, think $Q \simeq X/\hat{\mathbb{G}}_a$.]

Prove :

- Q geom. quotient $\Rightarrow$ $Q = \pi_0(X)$.
- EFFECTIVE CRITERIA for geometric quotient to be UNIVERSAL (stable under BC)

$\rightsquigarrow$

- EFFECTIVE CRITERIA for CATEGORICAL SIMPLICITY

INDEXED FRAMEWORK FOR DIFFERENTIAL GALOIS TH.

$$\begin{array}{l} \mathcal{A} \hookrightarrow \delta\text{-}\mathrm{Sch}_S \\ \quad \text{that admit } \pi_0 \\ \downarrow \pi_0 \\ \mathcal{X} = \mathrm{Sch}/_S \end{array}$$

- $\mathcal{P} : \mathcal{X}^{op} \longrightarrow \mathrm{CAT}$

$\rightsquigarrow \delta\text{-}\mathcal{P} : \mathcal{A}^{op} \longrightarrow \mathrm{CAT}$

- $C : \mathcal{P} \circ \pi_0 \Rightarrow \delta\text{-}\mathcal{P}$ pseudonatural

$C_Z : \mathcal{P}(\pi_0(Z)) \longrightarrow \delta\text{-}\mathcal{P}(Z)$ constant object functor

<u>Def</u>

$f : (X, \delta_X) \longrightarrow (Y, \delta_Y) \in \mathcal{A}$ is pre-PV for $\mathcal{P}$, if:

(1) f is effective descent for $\delta\text{-}\mathcal{P}$

(2) X, $X \times_Y X$, $X \times_Y X \times_Y X$ are simple for $\mathcal{P}$,

<u>Def</u>

class of scheme morphisms

If $U : \mathcal{P} \Rightarrow \mathcal{S}$ suitable,

f is PV for U, if:

(1) — " —

(2) X simple & $f \in \mathrm{Split}[f]$.

GALOIS THEORY OF DIFFERENTIAL SCHEMES

NB

- f PV $\Rightarrow$ f pre-PV
- f pre-PV $\Rightarrow$ $\mathbb{G}_f = \left(X \times_Y X \times_Y X \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; X \times_Y X \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X \right) \in \mathrm{Cat}(\mathcal{A})$

  $\Rightarrow$ $\mathrm{Gal}[f] = \pi_0(\mathbb{G}_f) \in \mathrm{PreCat}(\mathcal{X})$ GALOIS PRECATEGORY.
- f PV $\Rightarrow$ $\mathrm{Gal}[f] \in \mathrm{Cat}(\mathcal{X})$ GROUPOID.

Th

- f pre-PV $\Rightarrow$ equivalence

$$\mathrm{Split}_C[f] \simeq \mathcal{P}^{\mathrm{Gal}[f]}.$$

- f PV $\Rightarrow$ RHS: groupoid actions.

Pf f pre-PV $\Rightarrow$ $X,\ X \times_Y X,\ X \times_Y X \times_Y X$ simple $\Rightarrow$ $C_X,\ C_{X \times_Y X},\ C_{X \times_Y X \times_Y X}$ f.f.

$\Rightarrow$ Janelidze's indexed Galois Th. applies.

APPLICATIONS :

① QUASI-PROJECTIVE THEORY

- (K, δ_K) diff. field char 0, $k = \mathrm{Const}(K)$, $S = \mathrm{Spec}(k)$,
- $(X, \delta_X) \xrightarrow{f} (Y, \delta_Y) = \mathrm{Spec}(K, \delta_K)$ "classically simple"
- f is self-split.

$\Rightarrow$ f is PV, $G = \mathrm{Gal}[f]$ is an S-group scheme, correspondence :

Split quotients

$$\begin{tikzcd} & X \arrow[dl, "\text{fpqc}"'] \arrow[dd, "f"] \\ P \arrow[dr, "p"', "\text{quasi-proj.}"] & \\ & Y \end{tikzcd}$$

$\longleftrightarrow$

Closed subgroups

$G' \leq G$

$X/G' \longleftarrow G'$

$p \longmapsto \mathrm{Gal}[X \to P]$

$\rightsquigarrow$ unifies linear PV theory and STRONGLY NORMAL th. of KOLCHIN.

APPLICATIONS :

2 PARAMETRIC SYSTEMS of DIFFERENTIAL EQUATIONS

- over a scheme of parameters
- suitable for spectral problems, $\pi_0(X) = \Gamma \longleftarrow$ spectral curve

3 CANONICAL PV-GALOIS GROUPOID of a linear ODE

$$G_1 \rightrightarrows G_0$$

choices of PV extensions

isomorphisms between them.

[$\rightsquigarrow$ Deligne's Tannakian groupoid]

OUTLOOK

Differential Galois Theory = (precategorical) Descent + Categorical Galois Theory

⇝ difference PV-style Galois Theory [uses lax actions, with Prezado]

⇝ common generalisations: - δ-σ theory [Di Vizio-Hardouin-Wibmer]

- δ-δ theory [Cassidy-Singer]
- partial case
- Hasse-Schmidt derivatives

⇝ commuting differential operators, spectral curves, integrable systems

⇝ climbing HIGHER