Being a Union-splitting is Decidable in Modal Logic

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Outline

Modal logic

Decision problems

Union-splittings

Main result

Conclusion

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Modal logic

- Modal formulas:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Box\varphi, \quad p \in \mathsf{Prop}$$

- Normal modal logics: sets of modal formulas that contain propositional tautologies and the K axiom $\Box(p \to q) \to \Box p \to \Box q$, and are closed under modus ponens, uniform substitution, and necessitation.

Modal logic

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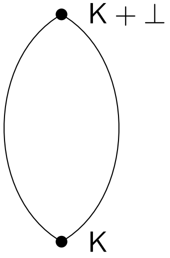
- $\mathsf{K} + \varphi$: the logic axiomatized by φ
- $\mathsf{NExt}\,\mathsf{K}$: the lattice of normal modal logics



NExt K

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Decidability of properties

"Does a modal logic L have a property P?"

- P: Kripke completeness, the finite model property, decidability, etc.

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- Classify properties by their complexity.

Decidability of properties

"Does a modal logic L have a property P?"

- P: Kripke completeness, the finite model property, decidability, etc.

"Is there an algorithm that decides whether a modal logic has the property P?"

- Classify properties by their complexity.
- Undecidability is a good excuse.

Decision problems of properties

Definition

Let P be a property. We say that P is decidable for normal modal logics if the set

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- Input: a formula φ, representing the logic $\mathsf{K} + \varphi$.
- "Decidability is undecidable."

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Is being a union-splitting decidable for normal modal logics?

- Answer: Yes!

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The idea of splittings originates in lattice theory, and was later applied to the lattice of equational theories of algebras, and then to the lattice of normal modal logics.

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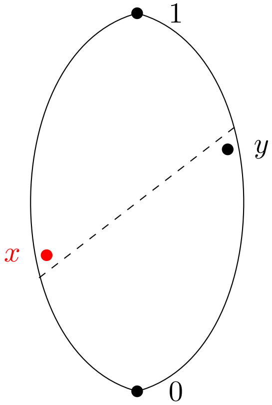
Definition

Let X be a complete lattice. A splitting pair of X is a pair (x, y) of elements of X such that $x \nleq y$ and for any $z \in X$, either $z \leq y$ or $x \leq z$. If (x, y) is a splitting pair of X, we say that y splits X and x is a splitting in X.

Union-splittings



Union-splittings

We apply this notion to the lattice $\mathsf{NExt}\,\mathsf{K}$.

- A logic L is a splitting in $\mathsf{NExt}\,\mathsf{K}$ if it is a splitting in the lattice-theoretic sense.
- A logic L is a union-splitting in $\mathsf{NExt}\,\mathsf{K}$ if it is a union of splittings.

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Union-splitting logics are related to Kripke completeness and axiomatization problems.

Union-splittings

Theorem (Blok, 1978; Chagrov)

For a normal modal logic L, the following are equivalent:

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Theorem (Blok, 1978; Chagrov)

For a normal modal logic L, the following are equivalent:

1. L *is a union-splitting in* NExt K *or the inconsistent logic,*
2. L *is strictly Kripke complete,*
   - *that is, for any logic $L' \neq L$, there exists a Kripke frame $\mathcal{F}$ such that $\mathcal{F} \models L$ and $\mathcal{F} \not\models L'$ or vice versa,*

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1. L *is a union-splitting in* $\mathsf{NExt}\,\mathsf{K}$ *or the inconsistent logic,*
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   - *that is, for any logic* $L' \neq L$*, there exists a Kripke frame* $\mathcal{F}$ *such that* $\mathcal{F} \models L$ *and* $\mathcal{F} \not\models L'$ *or vice versa,*
3. *The axiomatization problem for* L *is decidable,*
   - *that is, the property "$= L$" is decidable.*

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Decidability of being a union-splitting

Problem (Wolter & Zakharyaschev, 2007)

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Theorem (T.)

Being a union-splitting is decidable for normal modal logics.

Main Lemma

Lemma

For a modal logic L, the following are equivalent:

- *L is a union-splitting in* NExt K.
- *For any finite frame $\mathcal{F}$ such that $\mathcal{F} \not\models L$, there is a rooted cycle-free upset of $\mathcal{F}$ that refutes L.*

Main Lemma

Lemma

For a modal logic L, the following are equivalent:

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Example

- $\mathsf{KD} = \mathsf{K} + \Diamond\top$ is a union-splitting in $\mathsf{NExt}\,\mathsf{K}$.

Main Lemma

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Example

- $\mathsf{KD} = \mathsf{K} + \Diamond\top$ is a union-splitting in $\mathsf{NExt}\,\mathsf{K}$.
- $\mathsf{K} + \Box\Diamond\top$ is not a union-splitting in $\mathsf{NExt}\,\mathsf{K}$, witnessed by $\circ\!\!\longrightarrow\!\!\bullet$.

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Proof Sketch.

Given a formula φ, the algorithm runs the following two processes in parallel:

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Given a formula φ, the algorithm runs the following two processes in parallel:

- Try to verify that $\mathsf{K} + \varphi$ is a union-splitting.
  - This is semi-decidable via Jankov formulas.

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- Enumerate all finite frames $\mathcal{F}$ such that $\mathcal{F} \not\models \varphi$.
  - Check if all rooted cycle-free upsets of $\mathcal{F}$ validate φ.
  - If this is the case, then $\mathsf{K} + \varphi$ is not a union-splitting.

$\square$

Decidability of being a union-splitting

Corollary

The following properties are decidable for normal modal logics:

- *Strict Kripke completeness,*
- *Having a decidable axiomatization problem.*

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- There are continuum many union-splittings and non-union-splittings in NExt K.
- In each class, there are infinitely many finitely aximatizable ones.

Future work

- Computational complexity of being a union-splitting.

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- Computational complexity of being a union-splitting.
- Computability taking all logics into account.

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