Relative (co-)Yoneda Lemma for Categories Indexed over a Small-Generated Site

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Istituto Grothendieck, École Polytechnique

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Outline of the talk

1. Background and motivation

2. Preliminaries : Indexed categories and stacks

3. Indexed Yoneda lemma

4. Indexed (co)limits

1 Background and motivation

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What about relative category theory over a fixed arbitray topos ?

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Internal categories are too rigid in practice. For example, the canonical stack of a site is not an internal category !

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The indexed categories, although they have already been studied in this light (Paré, Schumacher), are yet to be fully understood (Pitts).

2 Preliminaries : Indexed categories and stacks

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We write $\mathrm{CAT}_{\mathcal{C}}$ for the 2-category of $\mathcal{C}$-indexed categories, functors and natural transformations between them.

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Stacks over a site

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To fix this problem, we must take into account the topology on $\mathcal{C}$.

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We write $\mathrm{St}(\mathcal{C}, J)$ for the full sub-2-category of $\mathrm{CAT}_{\mathcal{C}}$ on the J-stacks.

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- An indexed category can still be viewed locally as a category relative to a topos thanks to the associated stack 2-functor.

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which are defined componentwise in an obvious way using the corresponding ordinary constructions in CAT.

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- This extends to a 2-functor $-/C : \mathrm{CAT}_{\mathcal{C}} \to \mathrm{CAT}_{\mathcal{C}/C}$, called the localization 2-functor at C.

3 Indexed Yoneda lemma

Exponential indexed categories

Classically, under suitable assumptions on two categories $\mathcal{D}$ and $\mathcal{E}$, we can construct the category $\mathcal{E}^{\mathcal{D}}$ of functors from $\mathcal{D}$ to $\mathcal{E}$.

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Let $\mathcal{C}$ be a category and $\mathbb{D}$, $\mathbb{E}$ be two $\mathcal{C}$-indexed categories. We have a category $\mathrm{CAT}_{\mathcal{C}}(\mathbb{D}, \mathbb{E})$, but we want to construct a $\mathcal{C}$-indexed category $\mathbb{E}^{\mathbb{D}}$.

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Proposition (Paré and Schumacher)

Assume that $\mathbb{E}^{\mathbb{D}}$ is well-defined. Then we have pseudonatural equivalences of categories $\mathrm{CAT}_{\mathcal{C}}(\mathbb{F}, \mathbb{E}^{\mathbb{D}}) \cong \mathrm{CAT}_{\mathcal{C}}(\mathbb{F} \times \mathbb{D}, \mathbb{E})$.

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Theorem (Canonical stack of a site)

The $\mathcal{C}$-indexed category $\widehat{\mathbb{C}}_J$ is a J-stack.

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- For $C \in \mathcal{C}$ and $X \in \mathbb{D}(C)$, we have $よ_{\mathbb{D}_J, C}(X) : \mathbb{D}^\vee/C \Rightarrow \widehat{\mathbb{C}}_J/C$ whose component at $[d : D \to C] \in \mathcal{C}/C$ sends $Y \in \mathbb{D}(D)$ to $\mathbb{D}_{J,D}\left[Y, \mathbb{D}(d)(X)\right]$.

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Theorem (Indexed Yoneda lemma, Caramello and B.D.)

- *For* $C \in \mathcal{C}$, $X \in \mathbb{D}(C)$, $\mathbb{P} : \mathbb{D}^{\vee}/C \Rightarrow \widehat{\mathcal{C}/-}_{J\_}/C$ *and* $[d : D \to C] \in \mathcal{C}/C$, *there is a* *bijection* :

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which is *natural* *in all components ;*

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  whose components are given by the bijections described above.

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- Furthermore, our approach offers greater technical flexibility while being more computationally manageable.

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- A $\mathcal{C}$-indexed cocone to $\mathbb{F}$ is a global object $(X_C)_{C\in\mathcal{C}}$ of $\mathbb{E}$ together with "compatible" morphisms $(\Psi_{C,Y} : \mathbb{F}_C(Y) \to X_C)_{C\in\mathcal{C}, Y\in\mathbb{D}(C)}$ of $\mathbb{E}$;

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- A $\mathcal{C}$-indexed cocone to $\mathbb{F}$ is a global object $\mathbb{x} : 1_{\widehat{\mathcal{C}}} \Rightarrow \mathbb{E}$ of $\mathbb{E}$ together with a $\mathcal{C}$-indexed natural transformation $\Psi : \mathbb{F} \Rrightarrow \Delta(\mathbb{x})$;
- A $\mathcal{C}/C$-indexed cocone to $\mathbb{F}$, for $C \in \mathcal{C}$, is an object $X \in \mathbb{E}(C)$ together with a $\mathcal{C}/C$-indexed natural transformation $\Psi : \mathbb{F}/C \Rrightarrow \Delta(X)$;
- A $\mathcal{C}$-indexed cocone to $\mathbb{F}$ is a $\mathcal{C}$-indexed colimit if and only if it is universal under all localizations.

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Consequently, there is a priori no reason for the ordinary colimit of F to be its M-indexed colimit...

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Then any $\mathcal{C}$-indexed functor $\mathbb{F} : \mathbb{D} \Rightarrow \mathbb{E}$, where the fibers of $\mathbb{D}$ are small, admits a $\mathcal{C}$-indexed colimit whose component at $C \in \mathcal{C}$ is the colimit of :

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Interestingly, this allows for a unified and natural treatment of indexed limits and colimits, in contrast to the work of Paré and Schumacher.

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Let $(\mathcal{C}, J)$ be a site and $\mathbb{D}$ be a $\mathcal{C}$-indexed category such that $\widehat{\mathbb{D}_J} := \widehat{\mathbb{C}}_J^{\mathbb{D}^\vee}$ is well-defined. Let $\mathbb{P} : \mathbb{D}^\vee \Rightarrow \widehat{\mathbb{C}}_J$ be a $\mathcal{C}$-indexed functor.

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Theorem (Indexed co-Yoneda lemma, Caramello and B.D.)

The $\mathcal{C}$-indexed functor $\mathbb{P}$ is the $\mathcal{C}$-indexed colimit of $よ_{\mathbb{D}_J} \circ \pi_{\mathbb{P}}$, where $\pi_{\mathbb{P}}$ is the projection from the indexed category of elements of $\mathbb{P}$ to $\mathbb{D}$.

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- In particular, we now have sufficient groundwork to properly define the notion of indexed Grothendieck topology.

- We plan to introduce a formal type-based language in order to automate the relativization process as much as possible.

Thank you for your attention !

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