Relative Points and Completions of Indexed Categories

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Relative boolean algebras

Stone duality states $\mathbf{Bool}^{\mathrm{op}} \simeq \mathbf{Stone} \simeq \mathbf{Pro}(\mathbf{Fin})$.
The notions of Boolean algebra and of finite cardinal can be interpreted in any (Grothendieck) topos $\mathcal{E}$.

Question

Do we still have $\mathbf{Bool}(\mathcal{E})^{\mathrm{op}} \simeq \mathbf{Pro}(\mathbf{Fin}(\mathcal{E}))$ for an arbitrary Grothendieck topos $\mathcal{E}$?

The answer is no (even in a simple topos such as the topos of group actions of $\mathbf{Z}$ on sets).
To obtain this result, we replace the classical pro-completion by its relative analogue, as defined below.

From ordinary to relative toposes

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Topos of sets $\mathbf{Set}$	Sheaf topos $\mathbf{Sh}(\mathcal{C}, J)$
Category $\mathcal{D}$	
(Cofiltered) limits in $\mathcal{D}$	
Pro-completion $\mathbf{Pro}(\mathcal{D})$	
Presheaf topos $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$	
Cat. of pts $\mathbf{Pt}(\mathcal{E}) = \mathfrak{Top}(\mathbf{Set}, \mathcal{E})$	
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This correspondence answers the previous question:

- $\mathcal{C} = \mathbf{Z}$ (viewed as a one-object category) and $J^{\mathrm{triv}}_{\mathbf{Z}}$ the trivial topology;
- $\mathbf{Sh}(\mathcal{C}, J) = \mathbf{Set}^{\mathbf{Z}^{\mathrm{op}}}$;
- $\mathbb{D} = \mathbf{Fin}_{\mathbf{Z}}$ (the $\mathbf{Z}$-indexed category with value the category of finite sets and functions between them, with the trivial action of $\mathbf{Z}$);
- $\mathbf{Bool}(\mathbf{Set}^{\mathbf{Z}^{\mathrm{op}}})^{\mathrm{op}} \simeq \mathbf{Pro}_{\mathbf{Z}}(\mathbf{Fin}_{\mathbf{Z}})$ (the $\mathbf{Z}$-indexed pro-compl. of $\mathbf{Fin}_{\mathbf{Z}}$).

Indexed categories and stacks

Let $\mathcal{C}$ be a category.

Definitions

1. A $\mathcal{C}$-indexed category is a pseudo-functor $\mathbb{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{Cat}$.

1-dimensional (**Set**)	2-dimensional ($\mathfrak{Cat}$)
Presheaves $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ Natural transformations Category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ Sheaves	Indexed categories $\mathbb{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{Cat}$

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$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "\mathbb{D}_1"] \arrow[r, bend right, "\mathbb{D}_2"'] \arrow[r, phantom, "\Downarrow\Phi"] & \mathfrak{Cat} \end{tikzcd}.$$

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Natural transformations	Indexed functors
Category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$	2-category $\mathfrak{Cat}_{\mathcal{C}}$
Sheaves	

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Let $\mathcal{C}$ be a category endowed with a Grothendieck topology J.

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3. A J-stack is a $\mathcal{C}$-indexed category $\mathbb{D}$ such that for all J-covering sieves $\mathcal{R}$ on an object U of $\mathcal{C}$, $\mathbb{D}(U) \xrightarrow{\sim} \varprojlim_{(V \to U) \in \mathcal{R}} \mathbb{D}(V)$.

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Natural transformations	Indexed functors
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Sheaves	Stacks

Examples of indexed categories

- Constant $\mathcal{C}$-indexed category with value $\mathcal{D}$ defined by

$$\mathcal{D}_{\mathcal{C}}(V \xrightarrow{u} U) = \mathcal{D} \xrightarrow{1_{\mathcal{D}}} \mathcal{D} \quad \text{for all morphisms } V \xrightarrow{u} U \text{ of } \mathcal{C}.$$

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- For $\mathcal{D} = \mathbf{1}$, $\mathbf{1}_{\mathcal{C}}$ is the terminal $\mathcal{C}$-indexed category.
- Canonical stack of the topos $\mathbf{Sh}(\mathcal{C}, J)$

$$\mathbb{S}_{(\mathcal{C},J)}(U) = \mathbf{Sh}(\mathcal{C}_{/U}, J_{/U}) \simeq \mathbf{Sh}(\mathcal{C}, J)_{/U}$$

and for every $u \colon V \to U$,

$$\mathbb{S}_{(\mathcal{C},J)}(u) = u^* \colon \mathbf{Sh}(\mathcal{C}, J)_{/U} \longrightarrow \mathbf{Sh}(\mathcal{C}, J)_{/V}.$$

G-indexed categories

Let G be a group viewed as a one-object category.

- A G-indexed category $\mathbb{D}$ consists in a pseudo-action of G on a category, i.e.
  1. a category $\mathbb{D}(\bullet)$;
  2. a family of auto-equivalences $\left(\mathbb{D}(g) \colon \mathbb{D}(\bullet) \xrightarrow{\simeq} \mathbb{D}(\bullet)\right)_{g \in G}$ compatible with the group structure.
- For any category $\mathcal{D}$, the constant G-indexed category $\mathcal{D}_G$ corresponds to the trivial action of G of $\mathcal{D}$

$$\mathcal{D}_G(\bullet) = \mathcal{D} \quad \text{and} \quad \mathcal{D}_G(g) = 1_{\mathcal{D}} \quad \text{for all } g \in G.$$

- A G-indexed functor $\mathbf{1}_G \to \mathcal{D}_G$ corresponds to the datum of an object X of $\mathcal{D}$ with a right action of G on it, i.e. a family of automorphisms $(\sigma^g \colon X \xrightarrow{\simeq} X)$ that is compatible with the group structure.

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The ordinary pro-completion $\mathbf{Pro}(\mathcal{D})$ is constructed as the full subcategory of $(\mathbf{Set}^{\mathcal{D}})^{\mathrm{op}}$ formed by the pro-representable $F \colon \mathcal{D} \to \mathbf{Set}$, i.e. the small cofiltered limits of representable functors $\mathrm{Hom}(X, -) \colon \mathcal{D} \to \mathbf{Set}$.

Indexed limits (adapted from [ParSch78])

Recall that the limit of a diagram $\Phi\colon \mathcal{I} \to \mathcal{D}$ is a cone that is *universal*. In indexed category theory, we have the indexed analogue of it:

Definitions

Consider a $\mathcal{C}$-indexed functor $\Phi\colon \mathbb{I} \to \mathbb{D}$.

A $\mathcal{C}$-indexed cone over Φ consists in the data of:

(vertex) a $\mathcal{C}$-indexed functor $X\colon \mathbf{1}_{\mathcal{C}} \to \mathbb{D}$;

(legs) for every object U of $\mathcal{C}$, a cone $(\phi_I\colon X_U \to \Phi_U(I))$ over Φ_U in the fibre $\mathbb{D}(U)$

satisfying a compatibility condition with respect to base change by morphisms in $\mathcal{C}$.

A $\mathcal{C}$-indexed limit of Φ is a $\mathcal{C}$-indexed cone universal in a sense stable under base change.

An example of G-indexed limit

Let X be a Stone space on which G operates continuously via $(\sigma^g)_{g\in G}$.

- (X, σ^g) yields an indexed functor $\mathbf{1}_G \to \mathbf{Stone}_G$.
- In **Stone**, X is the co-filtered limit $\varprojlim D$ over $\mathcal{I}$, where $\mathcal{I}$ is the category whose objects are the pairs (D, d), $d\colon X \to D$ being a continuous map towards a finite discrete space D.
- If $\mathcal{I}$ is equipped with the trivial action of G, then the indexed limit of the above diagram (viewed as G-indexed in the trivial way) is $(X, 1_X)$.
- If $\mathcal{I}$ is equipped with the auto-equivalences

$$(X \xrightarrow{p} D) \longmapsto p \circ \sigma^g, \quad g \in G$$

the indexed limit is now (X, σ^g).

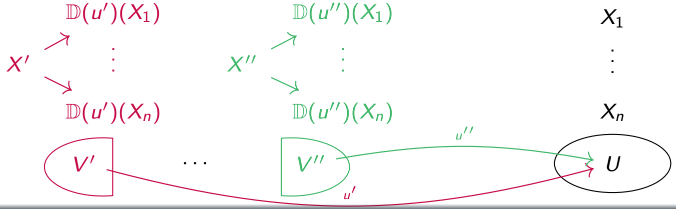
Locally cofiltered indexed categories

Let $(\mathcal{C}, J)$ be a site.

Definition ([SGA 4])

A $\mathcal{C}$-indexed category $\mathbb{D}$ is said to be locally cofiltered if:

- any finite (eventually empty) family of objects (X_i) in a fibre $\mathbb{D}(U)$ can be locally spanned;
- any parallel pair of arrows $Y \underset{g}{\overset{f}{\rightrightarrows}} X$ in a fibre $\mathbb{D}(U)$ can be locally equalised.



Indexed pro-completion

Definition

A $\mathcal{C}$-indexed functor $F\colon \mathbb{D} \to \mathbb{S}_{(\mathcal{C},J)}$ is called pro-representable if there are:

- a small locally cofiltered $\mathcal{C}$-indexed category $\mathbb{I}$;
- a $\mathcal{C}$-indexed functor $\Phi\colon \mathbb{I} \to \mathbb{D}$;

such that F is a $\mathcal{C}$-indexed limit of the $\underline{\mathrm{Hom}}(\Phi(I), -)$.

- Write $\mathbf{Pro}_{(\mathcal{C},J)}(\mathbb{D})$ for the full sub-category of $\mathfrak{Cat}_{\mathcal{C}}(\mathbb{D}, \mathbb{S}_{(\mathcal{C},J)})^{\mathrm{op}}$ formed by its pro-representable objects.
- $\mathbf{Pro}_{(\mathcal{C},J)}(\mathbb{D})$ is an ordinary category which can be made $\mathcal{C}$-indexed.

Definition ([CM])

The indexed pro-completion of $\mathbb{D}$ is the $\mathcal{C}$-indexed category

$$\underline{\mathbf{Pro}}_{(\mathcal{C},J)}(\mathbb{D})\colon U \longmapsto \mathbf{Pro}_{(\mathcal{C}_{/U},J_{/U})}(\mathbb{D}_{/U}).$$

Universal property

- $\underline{\mathbf{Pro}}_{(\mathcal{C},J)}(\mathbb{D})$ is a J-stack having all small locally cofiltered indexed limits.
- It is universal with this property:

Theorem ([CM])

For any J-stack $\mathbb{E}$ having all small locally cofiltered indexed limits, there is an equivalence between:

1. *the category of indexed functors $\mathbb{D} \to \mathbb{E}$;*
2. *the category of indexed functors $\underline{\mathbf{Pro}}_{(\mathcal{C},J)}(\mathbb{D}) \to \mathbb{E}$ preserving J-small locally cofiltered indexed limits.*

The definition of $\mathbf{Pro}_{(\mathcal{C},J)}(\mathbb{D})$ is algebraic, but it also admits a geometric description through relative points.

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Relative points

Definitions

Let $(\mathcal{C}, J)$ be a (small-generated) site.

1. A relative topos is a geometric morphism $p\colon \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$.
2. A relative morphism between relative toposes is
$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr, "q"'] & \arrow[d, phantom, "\overset{\sim}{\Leftarrow}"] & \mathcal{E} \arrow[dl, "p"] \\ & \mathbf{Sh}(\mathcal{C}, J) & \end{tikzcd}.$$
3. A relative point of $p\colon \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ is a relative morphism
$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}, J) \arrow[rr] \arrow[dr, "1_{\mathbf{Sh}(\mathcal{C},J)}"'] & \arrow[d, phantom, "\overset{\sim}{\Leftarrow}"] & \mathcal{E} \arrow[dl, "p"] \\ & \mathbf{Sh}(\mathcal{C}, J) & \end{tikzcd}.$$

When $\mathbf{Sh}(\mathcal{C}, J) = \mathbf{Set}$, relative toposes reduce to ordinary toposes.

Giraud toposes, i.e. relative presheaf toposes

Proposition ([CarZan21])

Let $(\mathcal{C}, J)$ be a (small-generated) site and $\mathbb{D}$ be a J-small $\mathcal{C}$-indexed category. Then the category of $\mathcal{C}$-indexed functors from $\mathbb{D}^{\mathrm{op}}$ to the canonical stack $\mathbb{S}_{(\mathcal{C},J)}$ is a Grothendieck topos

$$\mathrm{Gir}(\mathbb{D}) := \mathfrak{Cat}_{\mathcal{C}}(\mathbb{D}^{\mathrm{op}}, \mathbb{S}_{(\mathcal{C},J)}) \simeq \mathbf{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}}^{\mathrm{Gir}}),$$

where $\mathcal{G}(\mathbb{D})$ is the Grothendieck category of $\mathbb{D}$ and $J_{\mathbb{D}}^{\mathrm{Gir}}$ is the Giraud topology (i.e. the smallest topology making the projection functor $p_{\mathbb{D}} \colon \mathcal{G}(\mathbb{D}) \to \mathcal{C}$ a comorphism of sites).

- This topos is naturally defined over $\mathbf{Sh}(\mathcal{C}, J)$ by means of the structure morphism $\mathbf{Sh}(\mathcal{G}(\mathbb{D}), J^{Gir}) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by $p_{\mathbb{D}}$.
- $\mathrm{Gir}(\mathbb{D})$ thus constitutes the topos of "relative presheaves over $\mathbb{D}$".

Relative points of a Giraud topos

Let $\mathbb{D}$ be a J-small $\mathcal{C}$-indexed category.

- Any objet X of some $\mathbb{D}(U)$ yields a relative morphism $\mathbf{Sh}(\mathcal{C}, J)_{/U} \to \mathrm{Gir}(\mathbb{D})$ of inverse image the evaluation functor at X:

$$\mathrm{Gir}(\mathbb{D}) = \mathfrak{Cat}_{\mathcal{C}}(\mathbb{D}^{\mathrm{op}}, \mathbb{S}_{(\mathcal{C},J)}) \longrightarrow \mathbf{Sh}(\mathcal{C}, J)_{/U}.$$
$$(F \colon \mathbb{D}^{\mathrm{op}} \to \mathbb{S}_{(\mathcal{C},J)}) \longmapsto F(X)$$

- This defines an indexed functor

$$\mathbb{D} \longrightarrow \mathfrak{Top}_{/\,\mathbf{Sh}(\mathcal{C},J)}(\mathbf{Sh}(\mathcal{C}, J)_{/-}, \mathrm{Gir}(\mathbb{D})).$$

Theorem ([CM])

The previous indexed functor induces an equivalence

$$\mathbf{Pro}_{(\mathcal{C},J)}(\mathbb{D}) \xrightarrow{\sim} \mathbf{Pt}_{\mathbf{Sh}(\mathcal{C},J)}(\mathrm{Gir}(\mathbb{D}))^{\mathrm{op}}.$$

An example of Giraud topos

For $(\mathcal{C}, J) = (G, J_G^{\mathrm{triv}})$ and $\mathbb{D} = \mathbf{Fin}_G$, $\mathrm{Gir}(\mathbf{Fin}_G)$ has concretely:

1. for objects the presheaves $F\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Set}$ with a right action of G $(\phi^g\colon F \xrightarrow{\sim} F)_{g \in G}$;
2. for morphisms $(F_1, \phi_1) \to (F_2, \phi_2)$ the natural transformations $\tau\colon F_1 \Rightarrow F_2$ such that $\tau \circ \phi_1^g = \phi_2^g \circ \tau$ for every $g \in G$.

In fact, there is a bi-adjunction

$$\mathfrak{Cat}_G \underset{\mathfrak{Cat}_G(\mathbf{1}_G, -)}{\overset{(-)_G}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathfrak{Cat} \;,$$

so

$$\begin{aligned}
\mathrm{Gir}(\mathbf{Fin}_G) &= \mathfrak{Cat}_G(\mathbf{Fin}_G^{\mathrm{op}}, \mathbb{S}_{(G, J_G^{\mathrm{triv}})}) \\
&\simeq \mathfrak{Cat}(\mathbf{Fin}^{\mathrm{op}}, \mathbf{Set}^{G^{\mathrm{op}}}) && (\text{since } \mathfrak{Cat}_G(\mathbf{1}_G, \mathbb{S}_{(G, J_G^{\mathrm{triv}})}) \simeq \mathbf{Set}^{G^{\mathrm{op}}}) \\
&\simeq \mathbf{Set}^{\mathbf{Fin}^{\mathrm{op}}} \times \mathbf{Set}^{G^{\mathrm{op}}} && (\text{product of two presheaf toposes}).
\end{aligned}$$

Indexed pro-completion of finite G-sets

- First, $\mathbf{Pro}_{(G,J_G^{\mathrm{triv}})}(\mathbf{Fin}_G) \simeq \mathbf{Pt}_{\mathbf{Set}^{G^{\mathrm{op}}}}(\mathrm{Gir}(\mathbf{Fin}_G))^{\mathrm{op}}$.
- Moreover, since $\mathrm{Gir}(\mathbf{Fin}_G) \simeq \mathbf{Set}^{\mathbf{Fin}^{\mathrm{op}}} \times \mathbf{Set}^{G^{\mathrm{op}}}$ and $\mathbf{Set}^{\mathbf{Fin}^{\mathrm{op}}}$ classifies boolean algebras,

$$\mathbf{Pt}_{\mathbf{Set}^{G^{\mathrm{op}}}}(\mathrm{Gir}(\mathbf{Fin}_G)) \simeq \mathfrak{Top}(\mathbf{Set}^{G^{\mathrm{op}}}, \mathbf{Set}^{\mathbf{Fin}^{\mathrm{op}}}) \simeq \mathbf{Bool}(\mathbf{Set}^{G^{\mathrm{op}}}).$$

- Thus

$$\mathbf{Bool}(\mathbf{Set}^{G^{\mathrm{op}}})^{\mathrm{op}} \simeq \mathbf{Pro}_{(G,J_G^{\mathrm{triv}})}(\mathbf{Fin}_G).$$

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Karoubi envelope $\mathbf{Kar}(\mathcal{D})$	Ind. Karoubi envelope $\mathbf{Kar}_{(\mathcal{C},J)}(\mathbb{D})$
Cat of ess. pts $\mathbf{PtEss}(\mathcal{E})$	Cat. of rel. ess. pts $\mathbf{PtEss}_{\mathbf{Sh}(\mathcal{C},J)}(\mathcal{E})$
Irreducible objects	Relatively irreducible objects
$\mathbf{Kar}(\mathcal{D}) \simeq \mathbf{PtEss}(\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}})^{\mathrm{op}}$	$\mathbf{Kar}_{(\mathcal{C},J)}(\mathbb{D}) \simeq \mathbf{PtEss}_{\mathbf{Sh}(\mathcal{C},J)}(\mathrm{Gir}(\mathbb{D}))^{\mathrm{op}}$

KZ-monadicity

The pseudo-functor $\underline{\mathbf{Pro}}_{(\mathcal{C},J)} \colon \mathfrak{Cat}_{\mathcal{C}} \to \mathfrak{Cat}_{\mathcal{C}}$ defines a pseudo-monad on $\mathfrak{Cat}_{\mathcal{C}}$ with unit $L \colon 1_{\mathfrak{Cat}_{\mathcal{C}}} \Rightarrow \underline{\mathbf{Pro}}_{(\mathcal{C},J)}$.

Proposition ([CM])

$\underline{\mathbf{Pro}}_{(\mathcal{C},J)}$ *is a co-KZ-monad (also called a colax-idempotent pseudo-monad), i.e. there exists a modification*

$$\begin{tikzcd} \underline{\mathbf{Pro}}_{(\mathcal{C},J)} \arrow[r, shift left=2, "L_{\underline{\mathbf{Pro}}_{(\mathcal{C},J)}}", Rightarrow] \arrow[r, shift right=2, "\underline{\mathbf{Pro}}_{(\mathcal{C},J)}(L)"', Rightarrow] \arrow[r, phantom, "\Downarrow m"] & \underline{\mathbf{Pro}}_{(\mathcal{C},J)}\,\underline{\mathbf{Pro}}_{(\mathcal{C},J)} \end{tikzcd}$$

satisfying some coherence conditions.

To make idempotent this co-KZ-monad in a universal way, the strategy is to replace $\underline{\mathbf{Pro}}_{(\mathcal{C},J)}$ by the inverter of m.

Indexed Karoubi envelope

Proposition ([CM])

Let $\mathbb{D}$ be a $\mathcal{C}$-indexed category. The inverter of $m_{\mathbb{D}}$ is the full indexed sub-category $\underline{\mathbf{Kar}}_{(\mathcal{C},J)}(\mathbb{D})$ of $\underline{\mathbf{Pro}}_{(\mathcal{C},J)}(\mathbb{D})$ formed by the local retracts of indexed representable functors.

- Recall that a category $\mathcal{A}$ is said to be karoubian if any idempotent endomorphism $e\colon A \to A$ of $\mathcal{A}$ splits as $A \xrightarrow{p} B \xrightarrow{i} A$ with $p \circ i = 1_B$.
- $\underline{\mathbf{Kar}}_{(C,J)}(\mathbb{D})$ is the stack associated to the fiberwise Karoubi envelope of $\mathbb{D}$.

Theorem

For any J-stack $\mathbb{E}$ whose fibres are all karoubian, there is an equivalence between:

1. *the category of indexed functors $\mathbb{D} \to \mathbb{E}$;*
2. *the category of indexed functors $\underline{\mathbf{Kar}}_{(\mathcal{C},J)}(\mathbb{D}) \to \mathbb{E}$.*

Relative essential points

Proposition ([Bun79])

For any relative topos $p\colon \mathcal{E} \to \mathcal{F}$, there is a correspondence between:

- *relative essential points* $s\colon Id_{\mathbf{Sh}(\mathcal{C},J)} \to p$, *i.e. those such that* s^* *admits a* $\mathbf{Sh}(\mathcal{C},J)$*-indexed left adjoint* $s_!$;
- *relatively irreducible objects* X *of* $\mathcal{E}$, *i.e. those such that* $\underline{\mathrm{Hom}}_{\mathcal{E}}(X,-)$ *preserves indexed colimits.*

Theorem

Through the equivalence $\mathbf{Pro}_{(\mathcal{C},J)}(\mathbb{D}) \simeq \mathbf{Pt}_{\mathbf{Sh}(\mathcal{C},J)}(\mathrm{Gir}(\mathbb{D}))^{\mathrm{op}}$, *objects of* $\underline{\mathbf{Kar}}_{(\mathcal{C},J)}(\mathbb{D})$ *correspond to relative essential points of* $\mathrm{Gir}(\mathbb{D})$.

Stack associated to an indexed category

- Any $\mathcal{C}$-indexed category $\mathbb{D}$ can be turned into a stack.
- The construction can be performed by applying three times the pseudo-functor $(-)^+$ defined by

$$\mathbb{D}^+ : U \mapsto \varinjlim_{\mathcal{R} \in J(U)} \varprojlim_{(V \to U) \in \mathcal{R}} \mathbb{D}(V).$$

- We have a more geometric construction.

Proposition

The indexed sub-category of $\underline{\mathbf{Kar}}_{(\mathcal{C},J)}(\mathbb{D})$ formed by the locally representable objects is the stack associated to $\mathbb{D}$.

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Thank you for your attention!

Terminal category $\mathbf{1}$	Site $(\mathcal{C}, J)$
Topos of sets **Set**	Sheaf topos $\mathbf{Sh}(\mathcal{C}, J)$
Category $\mathcal{D}$	Indexed category $\mathbb{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{Cat}$
(Cofiltered) limits in $\mathcal{D}$	(Locally cofiltered) ind. limits in $\mathbb{D}$
Pro-completion $\mathbf{Pro}(\mathcal{D})$	Ind. pro-completion $\underline{\mathbf{Pro}}_{(\mathcal{C},J)}(\mathbb{D})$
Presheaf topos $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$	Giraud topos $\mathrm{Gir}(\mathbb{D})$
Cat. of pts $\mathbf{Pt}(\mathcal{E}) = \mathfrak{Top}(\mathbf{Set}, \mathcal{E})$	Cat. of rel. pts $\mathbf{Pt}_{/\,\mathbf{Sh}(\mathcal{C},J)}(\mathcal{E})$
$\mathbf{Pro}(\mathcal{D}) \simeq \mathbf{Pt}(\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}})^{\mathrm{op}}$	$\mathbf{Pro}_{(\mathcal{C},J)}(\mathbb{D}) \simeq \mathbf{Pt}_{\mathbf{Sh}(\mathcal{C},J)}(\mathrm{Gir}(\mathbb{D}))^{\mathrm{op}}$
Karoubi envelope $\mathbf{Kar}(\mathcal{D})$	Ind. Karoubi envelope $\underline{\mathbf{Kar}}_{(\mathcal{C},J)}(\mathbb{D})$
Cat of ess. pts $\mathbf{PtEss}(\mathcal{E})$	Cat. of rel. ess. pts $\mathbf{PtEss}_{\mathbf{Sh}(\mathcal{C},J)}(\mathcal{E})$
Irreducible objects	Relatively irreducible objects
$\mathbf{Kar}(\mathcal{D}) \simeq \mathbf{PtEss}(\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}})^{\mathrm{op}}$	$\mathbf{Kar}_{(\mathcal{C},J)}(\mathbb{D}) \simeq \mathbf{PtEss}_{\mathbf{Sh}(\mathcal{C},J)}(\mathrm{Gir}(\mathbb{D}))^{\mathrm{op}}$