Double-functorial representation of regular hyperdoctrines[1]

José Siqueira

Topos Institute & St Edmund's College, University of Cambridge

[1] https://arxiv.org/abs/2508.06637

(Some) relevant past work

Hermida (2000): the *bicategory* of spans and Beck-Chevalley;

Dawson, Paré, Pronk (2003-2010): the *double-categorical* angle on Beck-Chevalley;

Barwick (2017): adequate triples and the unfurling construction;

Aleiferi (2018): characterisation of double categories of spans;

Shulman (2008) and Moeller, Vasilakopoulou (2020): understanding of monoidal fibrations;

Paré (2024): companion commuters and conjoint commuters;

Nasu (2025): double-categorical regular logic from the fibrational perspective.

The big picture: functorial semantics for double categories

Goal: neatly separate the roles of the type-theoretical and logical operations by using double categories of contexts:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{context} \arrow[r, "\text{abstract operations on predicates}", "|" marking] \arrow[d, "\text{abstract operations on contexts}"'] \arrow[dr, phantom, "\text{compatibility constraints}"] & \text{context} \arrow[d, "\text{abstract operations on contexts}"] \\
\text{context} \arrow[r, "\text{abstract operations on predicates}"', "|" marking] & \text{context}
\end{tikzcd}$$

A semantics double functor then maps these abstract constraints to something concrete.

A double-categorical semantics allows for quantification and substitution to live in different 2-categories.

Double categories

A (weak) double category $\mathbb{C}$ is a pseudocategory internal to CAT. Concretely, there is a category $\mathbb{C}_0$ of objects and tight arrows, and a category $\mathbb{C}_1$ of loose arrows and squares, which are related by source and target functors, an identity assignment functor, and a composition functor, equipped with associativity and unitality constraints.

$$\begin{tikzcd}
A_1 \arrow[r, "|" marking] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & A_2 \arrow[r, "|" marking] \arrow[d, "g_1"] \arrow[dr, phantom, "\beta"] & A_3 \arrow[d, "h_1"] \\
B_1 \arrow[r, "|" marking] \arrow[d, "f_2"'] \arrow[dr, phantom, "\delta"] & B_2 \arrow[r, "|" marking] \arrow[d, "g_2"] \arrow[dr, phantom, "\gamma"] & B_3 \arrow[d, "h_2"] \\
C_1 \arrow[r, "|" marking] & C_2 \arrow[r, "|" marking] & C_3
\end{tikzcd}$$

Note: composition of loose arrows is not always strictly associative and unital, only composition of tight morphisms.

Double categories

A *companion* for a tight morphism $f\colon A \to B$ is a loose morphism $f^*\colon A \nrightarrow B$, together with squares

$$\begin{tikzcd}
A \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi_f"] & B \arrow[d, equals] \\
B \arrow[r, equals, "\shortmid"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equals, "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\psi_f"] & A \arrow[d, "f"] \\
A \arrow[r, "f^*"', "\shortmid"{marking}] & B
\end{tikzcd}$$

such that

$$\begin{tikzcd}
A \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "v_{f^*}"] & B \arrow[d, equals] \\
A \arrow[r, "f^*"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, equals, "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\psi_f"] & A \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi_f"] & B \arrow[d, equals] \\
A \arrow[r, "f^*"', "\shortmid"{marking}] & B \arrow[r, equals, "\shortmid"{marking}] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, equals, "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\
B \arrow[r, equals, "\shortmid"{marking}] & B
\end{tikzcd}
=
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A \arrow[r, equals, "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\psi_f"] & A \arrow[d, "f"] \\
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B \arrow[r, equals, "\shortmid"{marking}] & B
\end{tikzcd}.$$

Double categories

The dual notion is that of a *conjoint* $f_!$ of of a tight morphism f. These generalise adjunctions, but allows for the adjoints to live in different categories (e.g., doctrinal adjunctions).

Normal double functors (i.e., those preserving loose identities strictly) preserve companions and conjoints.

Companion commuters

A square

$$\begin{tikzcd} Y_1 \arrow[r, "Y", "|" marking] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & Y_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "X"', "|" marking] & X_2 \end{tikzcd}$$

in a double category $\mathbb{D}$ is a *companion commuter* if f_1 and f_2 have companions and its associated transpose square

$$\begin{tikzcd} Y_1 \arrow[r, equal, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\psi_{f_1}"] & Y_1 \arrow[r, "Y", "|" marking] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & Y_2 \arrow[r, "f_2^*", "|" marking] \arrow[d, "f_2"] \arrow[dr, phantom, "\varphi_{f_2}"] & X_2 \arrow[d, equal] \\ Y_1 \arrow[r, "f_1^*"', "|" marking] & X_1 \arrow[r, "X"', "|" marking] & X_2 \arrow[r, equal, "|" marking] & X_2 \end{tikzcd}$$

is an isomorphism in $\mathbb{D}_1$.

Double categories

We're interested in the following 'semantic' double categories:

- For any 2-category $\mathcal{K}$, the double category of quintets $\mathbb{Q}t(\mathcal{K})$ has the same objects as $\mathcal{K}$ and uses morphisms of $\mathcal{K}$ as both tight and loose arrows. Squares are just 2-cells in $\mathcal{K}$ (going up the page). A conjoint pair here is just an adjunction internal to $\mathcal{K}$.
- If T is a 2-monad on $\mathcal{K}$, the double category of algebras $\mathbb{A}lg(T;\mathcal{K})$ has T-algebras as objects, lax morphisms of T-algebras as tight arrows, colax morphisms of T-algebras as loose arrows, and certain compatible 2-cells as squares. A conjoint pair here is a doctrinal adjunction.

Recap: classical regular hyperdoctrines

Definition

A regular hyperdoctrine is a functor $P\colon \mathsf{C}^{\mathrm{op}} \to \mathbf{Pos}$ such that:

1. Each poset PX is $\wedge$-semilattice;
2. For each morphism $f\colon X \to Y$ in $\mathcal{C}$, the functor $Pf\colon PY \to PX$ has a left adjoint $\exists f$;
3. These adjoints satisfy the Beck-Chevalley condition: for any pullback square
$$\begin{array}{ccc} A & \xrightarrow{h} & I \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & J \end{array}$$
, the canonical map $\exists h \circ Pk \Rightarrow Pg \circ \exists f$ is invertible;
4. These adjoints satisfy the Fröbenius law: for each $f\colon X \to Y$, the canonical map $\exists f(Pf \wedge \mathrm{id}_{PX}) \Rightarrow \mathrm{id}_{PY} \wedge \exists f$ is invertible.

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- Quantification might not be possible along arbitrary terms.

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- Quantification might not be possible along arbitrary terms. Fix: consider adequate triples on contexts, compatible with the monoidal structure.

Adequate triples

Definition (adapted from Haugseng et al)

An *adequate triple* $\mathfrak{C} = (\mathsf{C}, \mathfrak{C}^l, \mathfrak{C}^r)$ consists of a category C, together with classes $\mathfrak{C}^l$ and $\mathfrak{C}^r$ of morphisms of C such that:

- $\mathfrak{C}^l$ and $\mathfrak{C}^r$ contain all identities and are closed under composition;
- Every cospan $X \xrightarrow{x} Z \xleftarrow{y} Y$ with $x \in \mathfrak{C}^l$ and $y \in \mathfrak{C}^r$ admits a pullback;
- The pullback of a morphism in $\mathfrak{C}^l$ along a morphism in $\mathfrak{C}^r$ is a morphism in $\mathfrak{C}^l$ (and vice-versa).

Adequate triples are objects for a cartesian 2-category $\mathcal{A}dq$, whose morphisms are the 2-functors that preserve $\mathfrak{C}^l$, $\mathfrak{C}^r$ and $\mathfrak{C}$-pullbacks, and whose 2-cells are the 2-natural transformations.

Examples of adequate triples

1. For any category C, there is a *trivial adequate triple* with underlying category C, where $\mathfrak{C}^{l} = \mathfrak{C}^{r} = \mathrm{mor}(\mathsf{C})$;
2. The category Met of metric spaces and continuous maps forms an adequate triple, where both the left and right class are the Lipschitz-continuous maps.
3. For any Grothendieck fibration $\pi\colon \mathsf{E} \to \mathsf{B}$, there is an associated adequate triple $\mathfrak{E}$ with underlying category E, where $\mathfrak{E}^{l}$ is the class of vertical maps and $\mathfrak{E}^{r}$ is the class of cartesian maps.
4. The category Top of compactly generated topological spaces admits an adequate triple structure, where the left class consists of Serre fibrations and the right class consists of arbitrary morphisms.

Spans

Adequate triples specify what is needed to compose spans:

$$\begin{tikzcd}
 & & X \odot X' \arrow[dl, "\chi"'] \arrow[dr, "\chi'"] \arrow[d, phantom, "\lrcorner"] & & \\
 & X \arrow[dl, "x_1"'] \arrow[dr, "x_2"] & {} & X' \arrow[dl, "x_2'"'] \arrow[dr, "x_3'"] & \\
X_1 & & X_2 & & X_3
\end{tikzcd}$$

There is a 2-functor $\mathcal{A}dq \to \mathrm{Dbl}$, which maps an adequate triple $\mathfrak{C}$ to its double category of spans $\mathbb{S}pan(\mathfrak{C})$: objects and tight arrows are as in C, loose arrows are spans, and squares are morphisms of spans.

Existential structures

Let $\mathfrak{C}$ be an adequate triple and $\mathcal{K}$ be a (locally small) 2-category. A $\mathfrak{C}$-existential $\mathcal{K}$-structure is a 2-functor $P\colon \mathsf{C}^{\mathrm{op}} \to \mathcal{K}$ such that:

- For each morphism $f\colon A \to B$ in $\mathfrak{C}^r$, the morphism $Pf\colon PB \to PA$ has a left adjoint $\exists^P f\colon PA \to PB$ (internal to $\mathcal{K}$).
- These adjoints satisfy the Beck-Chevalley condition with respect to the adequate triple $\mathfrak{C}$: for any $\mathfrak{C}$-pullback
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner\ \alpha"] & I \arrow[d, "g"] \\ B \arrow[r, "f"'] & J \end{tikzcd}$$
where $g \in \mathfrak{C}^l$ and $f \in \mathfrak{C}^r$, the 2-cell
$$\begin{tikzcd} PB \arrow[r, "\exists^P f"] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\eta^{Pf}"] & PJ \arrow[r, "Pg"] \arrow[d, "Pf"] \arrow[dr, phantom, "\Uparrow P(\alpha)"] & PI \arrow[r, equal] \arrow[d, "Ph"] \arrow[dr, phantom, "\Uparrow\varepsilon^{Ph}"] & PI \arrow[d, equal] \\ PB \arrow[r, equal] & PB \arrow[r, "Pk"'] & PA \arrow[r, "\exists^P h"'] & PI \end{tikzcd}$$
has an inverse $\mathcal{B}_\alpha$ in $\mathcal{K}$.

The 2-category of existential structures

The (cartesian) 2-category of existential structures and weak morphisms $\mathrm{ExistStr}_{\mathrm{w}}$ has:

- pairs $(\mathfrak{C}, P)$, where $\mathfrak{C}$ is an adequate triple and P is a $\mathfrak{C}$-existential structure as objects;
- morphisms $(\mathfrak{C}_1, P_1) \to (\mathfrak{C}_2, P_2)$ are triples (F, ρ, L) providing a diagram

$$\begin{tikzcd}
\mathsf{C}_1^{\mathrm{op}} \arrow[r, "P_1"] \arrow[d, "F^{\mathrm{op}}"'] \arrow[dr, phantom, "\Downarrow \rho"] & \mathcal{K}_1 \arrow[d, "L"] \\
\mathsf{C}_2^{\mathrm{op}} \arrow[r, "P_2"'] & \mathcal{K}_2
\end{tikzcd}$$

in $2\mathrm{CAT}$ for which F is a 1-morphism of adequate triples;

Existential structures

- 2-morphisms $\alpha\colon (F_1, \rho_1, L_1) \to (F_2, \rho_2, L_2)$ are pairs $(F_2 \xrightarrow{\alpha} F_1, L_1 \xrightarrow{\beta} L_2)$ of 2-natural transformations such that the cube

$$
\begin{tikzcd}
 & \mathsf{C}_1^{\mathrm{op}} \arrow[rr, "P_1"] \arrow[dl, "F_1^{\mathrm{op}}"'] \arrow[dd, equal] & & \mathcal{K}_1 \arrow[dl, "L_1"] \arrow[dd, equal] \\
\mathsf{C}_2^{\mathrm{op}} \arrow[rr, "P_2"'] \arrow[dd, equal] \arrow[r, phantom, "\Downarrow \alpha^{\mathrm{op}}"] & \arrow[r, phantom, "\overset{\rho_1}{\Leftarrow}"] & \mathcal{K}_2 \arrow[dd, equal] \arrow[r, phantom, "\Downarrow \beta"] & \\
 & \mathsf{C}_1^{\mathrm{op}} \arrow[rr, "P_1"] \arrow[dl, "F_2^{\mathrm{op}}"'] & & \mathcal{K}_1 \arrow[dl, "L_2"] \\
\mathsf{C}_2^{\mathrm{op}} \arrow[rr, "P_2"'] & \arrow[r, phantom, "\overset{\rho_2}{\Leftarrow}"] & \mathcal{K}_2 &
\end{tikzcd}
$$

commutes.

Connectives

Specifying a (symmetric) pseudomonoid in $\mathrm{ExistStr_w}$ amounts to selecting a (symmetric) monoidal adequate triple $\mathfrak{C}$, a locally small (symmetric) monoidal 2-category $\mathcal{K}$, and a $\mathfrak{C}$-existential $\mathcal{K}$-structure P whose underlying 2-functor is lax (symmetric) monoidal.

If $\mathfrak{C}$ gets the cartesian structure, this provides a monoidal structure on the fibres of P by the following result (a slight improvement on a result of Moeller & Vasilakopoulou):

Lemma

Let C be a cartesian category and $\mathcal{K}$ be a symmetric monoidal 2-category. Then there is an isomorphism between $\mathrm{SM}^{\mathrm{lax}}2\mathrm{Cat_{ps}}(\mathsf{C}^{\mathrm{op}}, \mathcal{K})$ and $2\mathrm{Cat_{ps}}(\mathsf{C}^{\mathrm{op}}, \mathrm{SM}(\mathcal{K}))$, where $2\mathrm{Cat_{ps}}(\mathcal{A}, \mathcal{B})$ is the 2-category of pseudofunctors, pseudonatural transformations, and modifications between $\mathcal{A}$ and $\mathcal{B}$.

$\mathfrak{C}$-regular hyperdoctrines

A $\mathfrak{C}$-*regular* $\mathcal{K}$-*hyperdoctrine* is a symmetric monoidal $\mathfrak{C}$-existential $\mathcal{K}$-structure for which the Frobënius law holds with respect to $\mathfrak{C}$: for each $f : A \to B$ in $\mathfrak{C}^r$, the canonical 2-cell

$$\begin{tikzcd}
PB \otimes PA \arrow[r, "\mathrm{id}_{PB} \otimes \exists f"] \arrow[d, "Pf \otimes \mathrm{id}_{PA}"'] & PB \otimes PB \arrow[d, equal] \\
PA \otimes PA \arrow[r, "\exists f \otimes \exists f"] \arrow[d, "\otimes_{PA}"'] \arrow[ur, phantom, "\Uparrow \varepsilon^f \otimes v_{\exists f}"] & PB \otimes PB \arrow[d, "\otimes_{PB}"] \\
PA \arrow[r, "\exists f"'] \arrow[ur, phantom, "\Uparrow \mathrm{mate}(\mu^{Pf})"] & PB
\end{tikzcd}$$

has an inverse $\mathfrak{F}^r_f$, where μ^{Pf} is the laxator for $Pf : (PB, \otimes_{PB}, I^{PB}) \to (PA, \otimes_{PA}, I^{PA})$.

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1. Every classical regular hyperdoctrine: take $\mathcal{K} = \mathrm{Pos}$ and give fibres the cartesian monoidal structure;

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2. Still with $\mathcal{K} = \mathrm{Pos}$, take $PA = [0, \infty]^A$, where $[0, \infty]$ is the Lawvere quantale, with the monoidal structure given by addition. For each $f : A \to B$, $\exists f : b \mapsto \inf_{a \in f^{-1}(b)}(\varphi(a))$ satisfies Beck-Chevalley and the monoidal form of Frobënius.

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1. Let C have finite limits and $PX = \mathsf{C} \downarrow X$, with the cartesian monoidal structure. Substitution and existential quantification are done by pullbacks and dependent sum as usual;
2. (Shulman 2011) If Top is the category of compactly generated topological spaces, then each slice $\mathsf{Top} \downarrow B$ is a model category and we can take $PB = \mathsf{Ho}(\mathsf{Top} \downarrow B)$, which is monoidal. This yields a $\mathfrak{C}$-regular Cat-hyperdoctrine, where $\mathfrak{C}$ is the adequate triple on Top that has the Serre fibrations as $\mathfrak{C}^l$, and arbitrary morphisms as $\mathfrak{C}^r$.

Hyperdoctrines as pseudo double functors

Lemma

There is a cartesian 2-functor $(-)^\bullet \colon \mathrm{ExistStr_w} \to \mathrm{Arrow}(\mathrm{Dbl})$, *where* Dbl *is the cartesian 2-category of (weak) double categories, pseudo double functors, and tight transformations.*

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As cartesian 2-functors preserve symmetric pseudomonoids, this means that every $\mathfrak{C}$-regular $\mathcal{K}$-hyperdoctrine P is mapped to a lax symmetric monoidal pseudo double functor between double categories.

The $(-)^\bullet$ construction

The pseudo double functor $P^\bullet\colon \mathbb{S}pan(\mathfrak{C})^{\mathrm{op}} \to \mathbb{Q}t(\mathcal{K})$ has tight component P, maps a $\mathfrak{C}$-span $X_1 \xleftarrow{x_1} X \xrightarrow{x_2} X_2$ to the composite $\exists^P x_2 \circ P x_1$ (as a loose arrow in $\mathbb{Q}t(\mathcal{K})$), and a square

$$\begin{tikzcd}
Y_1 \arrow[r, "Y", "|" marking] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & Y_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "X"', "|" marking] & X_2
\end{tikzcd}$$

in $\mathbb{S}\mathrm{pan}(\mathfrak{C})^{\mathrm{op}}$ to the square in $\mathbb{Q}t(\mathcal{K})$ corresponding to

$$\begin{tikzcd}
PY_1 \arrow[r, "P(y_1)"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & PY \arrow[r, "\exists y_2"] \arrow[d, equal] \arrow[r, phantom, shift right=5ex, "\Uparrow \eta^{y_2}"] & PY_2 \arrow[r, "P(f_2)"] \arrow[d, "P(y_2)"] \arrow[dr, phantom, "\circlearrowleft"] & PX_2 \arrow[r, equal] \arrow[d, "P(x_2)"] \arrow[r, phantom, shift right=5ex, "\Uparrow \varepsilon^{x_2}"] & PX_2 \arrow[d, equal] \\
PY_1 \arrow[r, "P(y_1)"] \arrow[d, "P(f_1)"'] \arrow[drr, phantom, "\circlearrowleft"] & PY \arrow[r, equal] & PY \arrow[r, "P(\alpha)"'] \arrow[d, "P(\alpha)"'] \arrow[drr, phantom, "\circlearrowleft"] & PX \arrow[r, "\exists x_2"'] & PX_2 \arrow[d, equal] \\
PX_1 \arrow[rr, "P(x_1)"'] & & PX \arrow[rr, "\exists x_2"'] & & PX_2
\end{tikzcd} .$$

The Beck-Chevalley cells are used to produce the loose compositor.

Hyperdoctrines as double functors

Theorem

Let $\mathfrak{C}$ be a cartesian adequate triple, $\mathcal{K}$ be a symmetric monoidal 2-category, and P be a $\mathfrak{C}$-regular $\mathcal{K}$-hyperdoctrine. Then the component $\mu^{\bullet}_{X,Y}$ of $P^{\bullet} \times P^{\bullet} \xrightarrow{\mu^{\bullet}} P^{\bullet}(- \times -)$ at $\mathfrak{C}$-spans $(X_1 \xleftarrow{x_1} X \xrightarrow{x_2} X_2, Y_1 \xleftarrow{y_1} Y \xrightarrow{y_2} Y_2)$ is a companion commuter square in $\mathbb{Q}t(\mathcal{K})$.

This is because we can build the companion commuter of $\mu^{\bullet}_{X,Y}$ out of the inverses of Frobënius cells and Beck-Chevalley cells for $\mathfrak{C}$-pullbacks (check the paper for a very satisfying string diagrams).

Double functors as hyperdoctrines

We also have the following converse result:

Theorem

Let $\mathfrak{C}$ be a cartesian adequate triple such that $\mathfrak{C}^l$ includes all diagonals, $\mathcal{K}$ be a symmetric monoidal 2-category, and $Q\colon \mathbb{S}pan(\mathfrak{C})^{\mathrm{op}} \to \mathbb{Q}t(\mathcal{K})$ be a lax symmetric monoidal double functor with companion commuter monoidal laxators. Then its tight component is a $\mathfrak{C}$-regular $\mathcal{K}$-hyperdoctrine.

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The idea is that $\mathbb{S}pan(\mathfrak{C})^{\mathrm{op}}$ has conjoints to all morphisms in the right class and companions to all the ones in the left class of $\mathfrak{C}$, which are preserved by (normal) double functors. This provides an existential quantifier satisfying Beck-Chevalley.

The companion commuter of the monoidal structure can then be used to construct the Frobënius 2-cells.

Discussion

Viewing hyperdoctrines as double functors facilitates studying their compositionality. The 2-category $\mathrm{ExistStr}_w$ can be seen as a sub-2-category of $\mathrm{Arrow}(\mathrm{Dbl})$, to which we can apply 2-categorical tools. For example, we can show it admits the construction of algebras.

We can consider a more general notion of regular double hyperdoctrine: contexts are given by a Beck-Chevalley double category with the necessary companions and conjoints (e.g., $\mathbb{S}pan(\mathsf{C})$, $\mathbb{R}el$, $\mathcal{V}Mat$), and semantics by appropriate double functors into a semantics double category (e.g., $\mathbb{A}lg(T;\mathcal{K})$).

Future work will investigate other adequate double-categories of contexts and semantics, as well as double-categorical triposes.

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