

# Quantale-Enriched Algebras: Exactness and Completeness

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TACL 2026

# Universal Algebra

Modern universal algebra was initiated by Birkhoff [Bir35].

Fix a signature  $\Sigma$  and a set  $V$  of variables.

- A  $\Sigma$ -algebra  $\underline{A}$  consists of a **set**  $A$  and, for each  $\sigma \in \Sigma(n)$ , an **operation**

$$\sigma^{\underline{A}} : A^n \longrightarrow A.$$

- A valuation  $\nu : V \rightarrow A$  extends uniquely to  $\bar{\nu} : \text{Term}_{\Sigma}(V) \rightarrow A$ .
- A **homomorphism**  $h : \underline{A} \rightarrow \underline{B}$  is an operation-preserving map  $h : A \rightarrow B$ .
- **Equations** are interpreted by equality:

$$\underline{A}, \nu \models s \approx t \iff \bar{\nu}(s) = \bar{\nu}(t).$$

# Equational Logic

Let  $E$  be a set of equations. The relation

$$E \vdash s \approx t$$

is generated by the following rules:

- **reflexivity**, **symmetry**, and **transitivity**:

$$\frac{}{s \approx s} \quad \frac{s \approx t}{t \approx s} \quad \frac{s \approx t \quad t \approx u}{s \approx u};$$

- the **congruence rule**, for every operation  $\sigma \in \Sigma(n)$ :

$$\frac{s_1 \approx t_1 \quad \dots \quad s_n \approx t_n}{\sigma(s_1, \dots, s_n) \approx \sigma(t_1, \dots, t_n)};$$

- substitution of terms for variables.

# Equational Completeness

Write

$$E \models s \approx t$$

when every  $\Sigma$ -algebra satisfying  $E$  also satisfies  $s \approx t$  under every valuation.

## Theorem 1 (Birkhoff's completeness theorem)

*For every set  $E$  of equations and all terms  $s, t \in \text{Term}_\Sigma(V)$ ,*

$$E \vdash s \approx t \iff E \models s \approx t.$$

# Partially-Ordered Algebras

The theory of partially-ordered algebras was developed by Bloom, Wright, and Pigozzi [Blo76, BW83, Pig04].

Fix a signature  $\Sigma$  and a set  $V$  of variables.

- A partially-ordered  $\Sigma$ -algebra  $\underline{A}$  consists of a **poset**  $(A, \leq)$  and, for each  $\sigma \in \Sigma(n)$ , a **monotone operation**

$$\sigma^{\underline{A}} : A^n \longrightarrow A.$$

- A valuation  $v : V \rightarrow A$  extends uniquely to  $\bar{v} : \text{Term}_{\Sigma}(V) \rightarrow A$ .
- A **homomorphism**  $h : \underline{A} \rightarrow \underline{B}$  is a monotone operation-preserving map  $h : A \rightarrow B$ .
- **Inequations** are interpreted by the order:

$$\underline{A}, v \models s \leq t \iff \bar{v}(s) \leq \bar{v}(t).$$

# Inequational Logic

Let  $E$  be a set of inequations. The relation

$$E \vdash s \leq t$$

is generated by the following rules:

- **reflexivity** and **transitivity**:

$$\frac{}{s \leq s} \quad \frac{s \leq t \quad t \leq u}{s \leq u};$$

- the **congruence rule**, for every monotone operation  $\sigma \in \Sigma(n)$ :

$$\frac{s_1 \leq t_1 \quad \dots \quad s_n \leq t_n}{\sigma(s_1, \dots, s_n) \leq \sigma(t_1, \dots, t_n)};$$

- substitution of terms for variables.

# Inequational Completeness

Write

$$E \models s \leq t$$

when every partially-ordered  $\Sigma$ -algebra satisfying  $E$  also satisfies  $s \leq t$  under every valuation.

## Theorem 2 (Completeness theorem for inequational logic)

*For every set  $E$  of inequations and all terms  $s, t \in \text{Term}_{\Sigma}(V)$ ,*

$$E \vdash s \leq t \iff E \models s \leq t.$$

In abstract algebraic logic, inequational logic is viewed as a 2-dimensional logical consequence relation, with ordered algebras as matrix models.

# Quantales

## Definition 3

A (commutative) *quantale* is a tuple

$$Q = (Q, \sqsubseteq, \bigsqcup, \cdot, e)$$

such that:

- $(Q, \sqsubseteq, \bigsqcup)$  is a complete join-semilattice;
- $(Q, \cdot, e)$  is a commutative monoid;
- multiplication preserves arbitrary joins in each variable:

$$q \cdot \bigsqcup_{i \in I} r_i = \bigsqcup_{i \in I} (q \cdot r_i).$$

## Basic examples

$$\mathbf{2} = (\{0 \leq 1\}, \leq, \vee, \wedge, 1), \quad \mathbb{L} = ([0, \infty], \geq, \inf, +, 0).$$

# Quantale-Enriched Categories

## Definition 4

A  $Q$ -category  $\mathbb{A}$  consists of a set  $\text{ob}\mathbb{A}$  and hom-values  $\mathbb{A}(a, a') \in Q$  satisfying

$$e \sqsubseteq \mathbb{A}(a, a), \quad \mathbb{A}(a, a') \cdot \mathbb{A}(a', a'') \sqsubseteq \mathbb{A}(a, a'').$$

## Definition 5

A  $Q$ -functor  $F : \mathbb{A} \rightarrow \mathbb{B}$  is a function on objects satisfying

$$\mathbb{A}(a, a') \sqsubseteq \mathbb{B}(Fa, Fa').$$

- **2**-categories are preorders; **2**-functors are monotone maps.
- $\mathbb{L}$ -categories are generalized metric spaces;  $\mathbb{L}$ -functors are nonexpanding maps.

# Blueprint: Quantale-Enriched Algebras

|                            | Carrier       | Operations                                                   | Morphisms              | Judgements     | Semantics                                          |
|----------------------------|---------------|--------------------------------------------------------------|------------------------|----------------|----------------------------------------------------|
| Algebras                   | set           | $A^n \rightarrow A$                                          | homomorphisms          | $s \approx t$  | $\bar{v}(s) = \bar{v}(t)$                          |
| Partially-ordered algebras | poset         | monotone $A^n \rightarrow A$                                 | monotone homomorphisms | $s \leq t$     | $\bar{v}(s) \leq \bar{v}(t)$                       |
| $Q$ -enriched algebras     | $Q$ -category | $Q$ -functor $\mathbb{A}^{\otimes n} \rightarrow \mathbb{A}$ | $Q$ -homomorphisms     | $s \vdash_q t$ | $q \sqsubseteq \mathbb{A}(\bar{v}(s), \bar{v}(t))$ |

Intuitively,  $s \vdash_q t$  means that  $s$  entails  $t$  to degree  $q$ .

$$\frac{}{s \vdash_e s} \quad \text{Reflexivity} \qquad \frac{s \vdash_q t \quad t \vdash_r u}{s \vdash_{q \cdot r} u} \quad \text{Transitivity.}$$

Our framework is relevant to the theory of quantitative algebras developed by Mardare, Panangaden, and Plotkin [MPP16].

# What's Next?

In **Set**, we know:

- what an equivalence relation is;
- what a quotient is.

What happens after enrichment?

- What is the appropriate notion of an equivalence relation (resp. congruence) on a  $Q$ -category (resp.  $Q$ -algebra)?
- What is the appropriate notion of quotient of a  $Q$ -category (resp.  $Q$ -algebra)?

More philosophically:

- What is a “congruence” in general?
- What is a “quotient” in general?

## Exactness

To answer these questions, we need an **exactness theory for  $Q$ -categories and  $Q$ -algebras!**

# Enriched Categories

Following Kelly [Kel82], enriched category theory replaces hom-sets by hom-objects in a symmetric monoidal category.

Let  $(\mathcal{V}, \otimes, I)$  be a symmetric monoidal category. A  $\mathcal{V}$ -category  $\mathcal{C}$  consists of:

- a collection of objects;
- a hom-object  $\mathcal{C}(X, Y) \in \mathcal{V}$  for each  $X, Y$ ;
- a composition law and identity morphisms in  $\mathcal{V}$ :

$$\mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z), \quad I \longrightarrow \mathcal{C}(X, X).$$

## The hierarchy of enriching categories

- **(Set,  $\times$ , 1)** is symmetric monoidal: **Set**-categories are ordinary categories.
- **(Pos,  $\times$ , 1)** is symmetric monoidal: **Pos**-categories are poset-enriched categories.

# Enrichment over $Q\text{-Cat}$

$Q$ -categories and  $Q$ -functors form a **symmetric monoidal closed category**

$$(Q\text{-Cat}_o, \otimes, [-, -], \mathbf{1}),$$

where

$$(\mathbb{A} \otimes \mathbb{B})((a, b), (a', b')) = \mathbb{A}(a, a') \cdot \mathbb{B}(b, b')$$

and the internal hom has  $Q$ -functors as objects, with

$$[\mathbb{A}, \mathbb{B}](F, G) = \bigwedge_{a \in \text{ob } \mathbb{A}} \mathbb{B}(Fa, Ga).$$

## Categories enriched over $Q\text{-Cat}$

We may therefore consider  $Q\text{-Cat}$ -categories  $\mathcal{C}$ : each hom is a hom- $Q$ -category  $\mathcal{C}(X, Y)$ , and composition is a  $Q$ -functor

$$\mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z).$$

In particular,  $Q\text{-Cat}$  is enriched over itself.

# Exactness in **Set** and **Alg**( $\Sigma$ )

Sets and functions form the category **Set**;  $\Sigma$ -algebras and homomorphisms form the category **Alg**( $\Sigma$ ).

## Sets

The kernel pair of  $f : A \rightarrow B$  is its **pullback** along itself:

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B. \end{array}$$

For an equivalence relation  $R \xrightarrow[r_2]{r_1} A$ , its quotient is the **coequalizer**  $q_R : A \rightarrow A/R$ .

## Algebras

The kernel pair of a homomorphism  $h : \underline{A} \rightarrow \underline{B}$  is its **pullback** along itself:

$$\begin{array}{ccc} \underline{A} \times_{\underline{B}} \underline{A} & \xrightarrow{\pi_2} & \underline{A} \\ \pi_1 \downarrow & & \downarrow h \\ \underline{A} & \xrightarrow{h} & \underline{B}. \end{array}$$

Its underlying equivalence relation is a congruence. A congruence  $R \subseteq A \times A$  is a subalgebra  $\underline{R}$  of  $\underline{A} \times \underline{A}$ . Its quotient is the **coequalizer**

$$\underline{R} \xrightarrow[r_2]{r_1} \underline{A} \xrightarrow{q_R} \underline{A}/R.$$

## Exactness principle

Every **equivalence relation** (resp. congruence) is the kernel pair of its coequalizer. Every **surjection** (resp. surjective homomorphism) is the coequalizer of its kernel pair.

# Exactness in **Pos** and **Alg**<sub>≤</sub>(Σ)

Both **Pos** and **Alg**<sub>≤</sub>(Σ) are **poset-enriched categories**.

In any poset-enriched category  $\mathcal{C}$ , the **comma object**  $f \downarrow f$  of a morphism  $f : A \rightarrow B$ , when it exists, is characterized by

$$\begin{array}{ccc} f \downarrow f & \xrightarrow{p_2} & A \\ p_1 \downarrow & \leq & \downarrow f \\ A & \xrightarrow{f} & B. \end{array}$$

In **Pos** and **Alg**<sub>≤</sub>(Σ), this comma object is computed as

$$f \downarrow f = \{(a, a') \in A \times A \mid fa \leq_B fa'\},$$

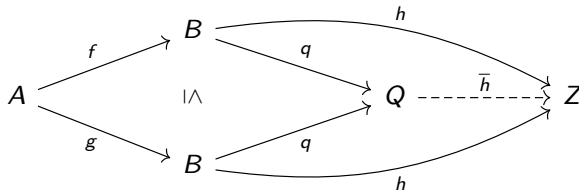
with the componentwise order

$$(a, a') \leq (b, b') \iff a \leq_A b \text{ and } a' \leq_A b'.$$

In **Alg**<sub>≤</sub>(Σ), the operations are inherited componentwise from  $A$ :

$$\sigma((a_1, a'_1), \dots, (a_n, a'_n)) = (\sigma(a_1, \dots, a_n), \sigma(a'_1, \dots, a'_n)).$$

In any poset-enriched category  $\mathcal{C}$ , the **coinserter** of  $f, g : A \rightarrow B$ , when it exists, is a morphism  $q : B \rightarrow Q$  such that  $qf \leq qg$  and, for every  $h : B \rightarrow Z$  with  $hf \leq hg$ , there is a unique  $\bar{h} : Q \rightarrow Z$  with  $\bar{h}q = h$ :



The coinserter  $q : B \rightarrow Q$  of  $f, g$  in **Pos** and  $\mathbf{Alg}_{\leq}(\Sigma)$  is computed as follows. In **Pos**, let  $\preceq_{f,g}$  be the least preorder on  $B$  containing  $\leq_B$  and all pairs  $(f(a), g(a))$ . In  $\mathbf{Alg}_{\leq}(\Sigma)$ , take the least such preorder that is also compatible with every operation.

$$b \sim b' \iff b \preceq_{f,g} b' \text{ and } b' \preceq_{f,g} b, \quad Q = B/\sim, \quad q(b) = [b],$$

with

$$[b] \leq_Q [b'] \iff b \preceq_{f,g} b'.$$

In  $\mathbf{Alg}_{\leq}(\Sigma)$ , the operations are inherited by

$$\sigma([b_1], \dots, [b_n]) = [\sigma(b_1, \dots, b_n)].$$

## Exactness principle

Every **preorder containing the order** (resp. compatible preorder containing the order) is the comma object of its coinserter. Every **surjective monotone map** (resp. surjective homomorphism) is the coinserter of its comma object.

# Relational Presheaves

A (continuous) relational presheaf on a set  $A$  is a  $Q$ -indexed family  $R = (R_q)_{q \in Q}$  of binary relations satisfying

$$q \sqsubseteq r \implies R_r \subseteq R_q, \quad \text{Id}_A \subseteq R_e, \quad R_q ; R_r \subseteq R_{q \cdot r},$$

together with the continuity condition

$$R_{\bigsqcup_{i \in I} q_i} = \bigcap_{i \in I} R_{q_i}.$$

## Theorem 6

*Relational presheaves and relational morphisms are equivalent to  $Q$ -categories and  $Q$ -functors:*

$$Q\text{-Cat}_o \simeq \mathbf{RelP}.$$

The correspondence is given by

$$R_q = \{(a, a') \mid q \sqsubseteq \mathbb{A}(a, a')\}, \quad \mathbb{A}(a, a') = \bigsqcup \{q \in Q \mid (a, a') \in R_q\}.$$

# The $Q$ -Kernel in $Q\text{-Cat}$

In  $Q\text{-Cat}$ , let  $f : \mathbb{A} \rightarrow \mathbb{B}$  be a  $Q$ -functor.

## Definition 7

The  $Q$ -kernel of  $f$  in  $Q\text{-Cat}$  is the relational presheaf  $\ker f$  on  $\text{ob}\mathbb{A}$  defined by

$$(\ker f)_q = \{(a, a') \in \text{ob}\mathbb{A} \times \text{ob}\mathbb{A} \mid q \sqsubseteq \mathbb{B}(fa, fa')\} \quad (q \in Q).$$

For each  $q \in Q$ , the relation  $(\ker f)_q$  is the object set of the  $q$ -comma object

$$\ker_q f = \text{comm}_q(f, f).$$

Hence each component is computed by a weighted limit, although the  $Q$ -kernel itself is not a weighted limit.

# The $Q$ -Coinserter

Let  $\mathcal{D}$  be a  $Q$ -**Cat**-category. For parallel morphisms  $u, v : X \rightarrow Y$ , write  $u \sqsubseteq_q v$  when  $q \sqsubseteq \mathcal{D}(X, Y)(u, v)$ .

## Definition 8

For each  $q \in Q$ , let  $d_q^0, d_q^1 : X_q \rightarrow A$  be morphisms in  $\mathcal{D}$ . Their  $Q$ -coinserter is a morphism  $c : A \rightarrow C$  satisfying:

- $cd_q^0 \sqsubseteq_q cd_q^1$  for every  $q \in Q$ ;
- every morphism  $g : A \rightarrow Z$  satisfying  $gd_q^0 \sqsubseteq_q gd_q^1$  for every  $q \in Q$  factors uniquely as  $g = hc$  for a morphism  $h : C \rightarrow Z$ ;
- for any such  $g, g'$ , with induced  $h, h'$ ,

$$\mathcal{D}(A, Z)(g, g') = \mathcal{D}(C, Z)(h, h').$$

# Computing $Q$ -Coinserter in $Q\text{-Cat}$

Let  $d_q^0, d_q^1 : \mathbb{X}_q \rightarrow \mathbb{A}$  be  $Q$ -functors for  $q \in Q$ .

**Step 1: adjoin the required edges.** Enlarge the underlying  $Q$ -graph of  $\mathbb{A}$  to

$$M(a, a') = \mathbb{A}(a, a') \sqcup \bigsqcup \{q \in Q \mid \exists x \in \text{ob} \mathbb{X}_q : d_q^0 x = a, d_q^1 x = a'\}.$$

Then  $q \sqsubseteq M(d_q^0 x, d_q^1 x)$  for every  $q \in Q$  and  $x \in \text{ob} \mathbb{X}_q$ .

**Step 2: close under finite paths.** Let  $\mathbb{C}$  be the free  $Q$ -category generated by  $M$ :

$$\mathbb{C}(a, a') = \bigsqcup_{n \geq 0} \bigsqcup_{a=a_0, \dots, a_n=a'} \prod_{i=1}^n M(a_{i-1}, a_i),$$

where the empty product is  $e$ .

We write  $\mathbb{C} = \text{coins}((d_q^0, d_q^1)_{q \in Q})$ . The identity-on-objects  $Q$ -functor  $c : \mathbb{A} \rightarrow \mathbb{C}$  is the corresponding  $Q$ -coinserter.

# Q-Relations and Quotients

Fix a  $Q$ -category  $\mathbb{A}$ .

## Definition 9

A  $Q$ -relation on  $\mathbb{A}$  is a  $Q$ -indexed family  $R = (R_q)_{q \in Q}$  of binary relations

$$R_q \subseteq \text{ob}\mathbb{A} \times \text{ob}\mathbb{A}.$$

No relational-presheaf conditions are imposed. The poset of  $Q$ -relations, ordered pointwise by inclusion, is denoted  $Q\text{-}\mathbf{Rel}(\mathbb{A})$ .

## Definition 10

A *quotient* of  $\mathbb{A}$  is a  $Q$ -functor  $f : \mathbb{A} \rightarrow \mathbb{B}$  that is bijective on objects. The isomorphism classes of such quotients over  $\mathbb{A}$ , ordered by factorization, form the poset  $Q\text{-}\mathbf{Quot}(\mathbb{A})$ .

# From $Q$ -Relations to Quotients

Fix  $R \in Q\text{-Rel}(\mathbb{A})$ . For each  $q \in Q$ , the projections

$$p_q^0, p_q^1 : R_q \longrightarrow \text{ob}\mathbb{A}$$

induce  $Q$ -functors from the discrete  $Q$ -category on  $R_q$ :

$$p_q^0, p_q^1 : DR_q \longrightarrow \mathbb{A}.$$

## Definition 11

The *quotient generated by  $R$*  is the  $Q$ -coinserter

$$\text{quot}R : \mathbb{A} \longrightarrow \mathbb{A}/R := \text{coins}((p_q^0, p_q^1)_{q \in Q}).$$

The  $Q$ -functor  $\text{quot}R$  is the identity on objects, hence bijective on objects, and freely forces

$$q \sqsubseteq (\mathbb{A}/R)(a, a') \quad \text{for every } (a, a') \in R_q.$$

# The Kernel–Quotient Adjunction

## Theorem 12 (Kernel–quotient adjunction)

*The maps  $\text{quot} : Q\text{-}\mathbf{Rel}(\mathbb{A}) \rightleftarrows Q\text{-}\mathbf{Quot}(\mathbb{A}) : \text{ker}$  form an adjunction  $\text{quot} \dashv \text{ker}$ .*

## Corollary 13

*$\text{ker} \circ \text{quot}$  is a closure operator.*

## Theorem 14 (Bridge lemma)

*$(\text{ker} \circ \text{quot})R$  is the smallest relational presheaf on  $\mathbb{A}$  containing  $R$ .*

## Theorem 15 (Exactness for $Q$ -categories)

*For every quotient  $f : \mathbb{A} \rightarrow \mathbb{B}$  and every relational presheaf  $R$  on  $\mathbb{A}$ ,*

$$f \cong \text{quot}(\text{ker}f), \quad R = \text{ker}(\text{quot}R).$$

# Q-Congruences

Let  $\underline{\mathbb{A}}$  be a  $Q$ -algebra with underlying  $Q$ -category  $\mathbb{A}$ .

## Definition 16

A  $Q$ -congruence on  $\underline{\mathbb{A}}$  is a relational presheaf  $R = (R_q)_{q \in Q}$  on  $\mathbb{A}$  compatible with every operation: for each  $\sigma \in \Sigma(n)$ ,

$$(a_i, b_i) \in R_{q_i} \quad (1 \leq i \leq n)$$

implies

$$(\sigma^{\underline{\mathbb{A}}}(a_1, \dots, a_n), \sigma^{\underline{\mathbb{A}}}(b_1, \dots, b_n)) \in R_{q_1 \dots q_n}.$$

For a nullary operation, the resulting degree is  $e$ .

The kernel-quotient adjunction and bridge lemma above restrict to  $Q$ -congruences and quotient  $Q$ -homomorphisms, yielding the corresponding exactness theorem for  $Q$ -algebras.

# Q-Inequational Logic QL

## Relaxation

$$\frac{t \vdash_q t'}{t \vdash_{q'} t'} \quad (q' \sqsubseteq q)$$

## Reflexivity

$$\overline{t \vdash_e t}$$

## Transitivity

$$\frac{t \vdash_q t' \quad t' \vdash_r t''}{t \vdash_{q \cdot r} t''}$$

## Continuity

$$\frac{(t \vdash_{q_i} t')_{i \in I}}{t \vdash_{\bigsqcup_{i \in I} q_i} t'}$$

## Congruence ( $\sigma \in \Sigma(n)$ )

$$\frac{(t_i \vdash_{q_i} t'_i)_{1 \leq i \leq n}}{\sigma(t_1, \dots, t_n) \vdash_{q_1 \dots q_n} \sigma(t'_1, \dots, t'_n)}$$

## Substitution ( $\tau : V \rightarrow \text{Term}_\Sigma(V)$ )

$$\frac{t \vdash_q t'}{\overline{\tau}(t) \vdash_q \overline{\tau}(t')}$$

For a set  $E$  of Q-inequations, write  $E \vdash_{\text{QL}} t \vdash_q t'$  when the judgement is generated from  $E$  by these six rules.

# Birkhoff-Style Completeness

## Theorem 17 (Completeness for QL)

For every set  $E$  of  $Q$ -inequations, all  $t, t' \in \text{Term}_\Sigma(V)$ , and every  $q \in Q$ ,

$$E \models t \vdash_q t' \iff E \vdash_{\text{QL}} t \vdash_q t'.$$

Let  $\overline{E}$  be the least fully invariant  $Q$ -congruence on  $\text{Term}_\Sigma(V)$  containing  $E$ , i.e. the least one closed under substitutions. Then

$$\underbrace{E \models t \vdash_q t'}_{\text{semantic consequence}} \iff (t, t') \in \overline{E}_q \iff \underbrace{E \vdash_{\text{QL}} t \vdash_q t'}_{\text{derivability}}.$$

- The first equivalence follows from kernel–quotient exactness.
- The second says that the six rules generate precisely  $\overline{E}$ .

# An Invitation to Quantale-Enriched Algebras

This framework is fully developed in my ongoing PhD thesis.

- a uniform refactoring of Wechler's classical proof strategy [Wec92] across the three settings;
- polarity and  $p$ -norm presentations;
- image and quotient–kernel factorization systems;
- Pavelka-style completeness;
- SP and HSP definability theorems;
- a ternary factorization system and transport to separated  $Q$ -algebras.

Quantale-Enriched Universal Algebra

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- ① Abstract algebraic logic and fuzzy logical consequence [FJP03, Bě15]
- ② Algebraic theories and metric monads [Adá22, ADV23, RT26]
- ③ Many-valued modal logic [BKPV13, KPT24]
- ④ Recasting important examples as  $Q$ -algebras [Cha58, GJKO07, HN18]

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Thank You!