A Duality for Algebras of Fractions: The Category $\mathbb{P}$ of Primed Posets

Brendan Dufty
University of Denver

Road Map

Part I

1. What is an algebra of fractions?
2. Why do we need a new duality?

Constructing the Duality

1. Irreducibles have a strong internal structure.
2. A closure operator resolves non-distributivity.
3. A morphism miracle.
4. The duality result.

Why Algebras of Fractions?

A Generalized Pattern

$$\text{cancellative comm. monoid} \xrightarrow{a \cdot b^{-1}} \text{Abelian group of fractions}$$

$$\text{bounded distributive lattice} \xrightarrow{\bigvee_i (a_i \wedge \neg b_i)} \text{free Boolean extension}$$

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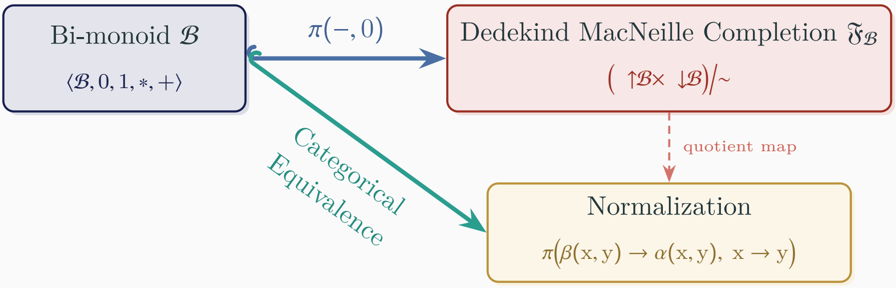
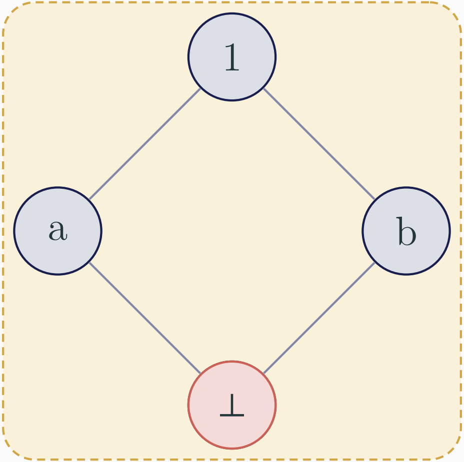
$$\text{bi-monoid} \xrightarrow{a \cdot {\sim} b} \text{algebra of fractions}$$

Galatos–Přenosil '23

The Fraction Recipe



$[\mathfrak{p}, 1]$ is Boolean



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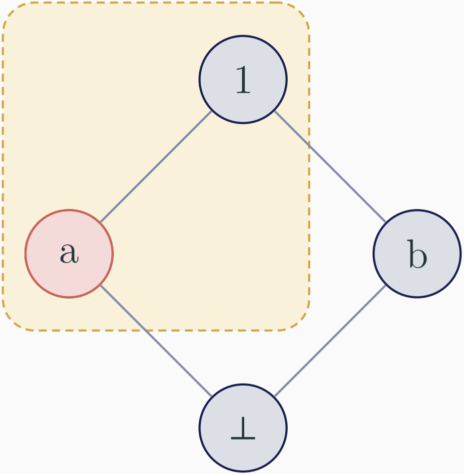

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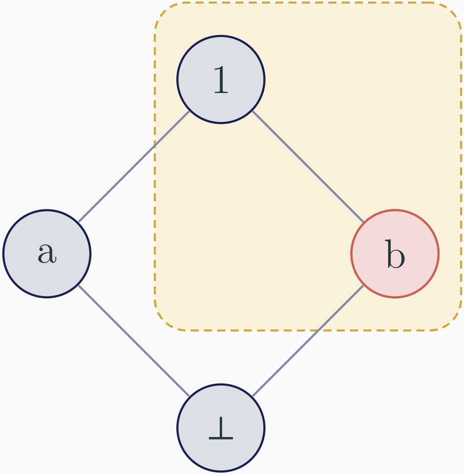

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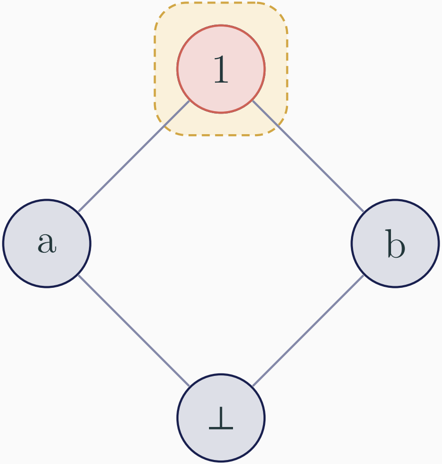

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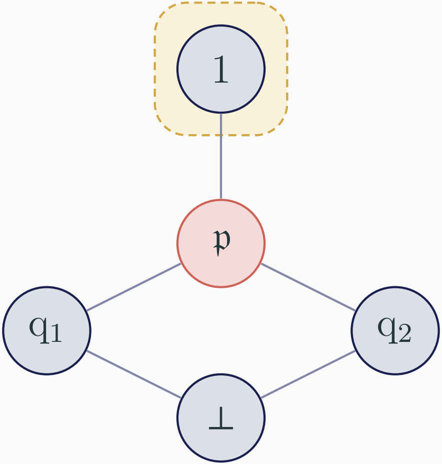


Uncovering Bimonoids

$$\text{Pointed Brouwerian Algebra } (\mathcal{B}, \mathfrak{p}) \xrightarrow{\text{extract}} \text{Bi-monoid } \mathcal{B}_{\mathrm{bi}} \xrightarrow{\text{completion}} \text{Algebra of Fractions } \mathfrak{F}_{\mathcal{B}}$$

Multiplication

$$a * b = a \wedge b$$

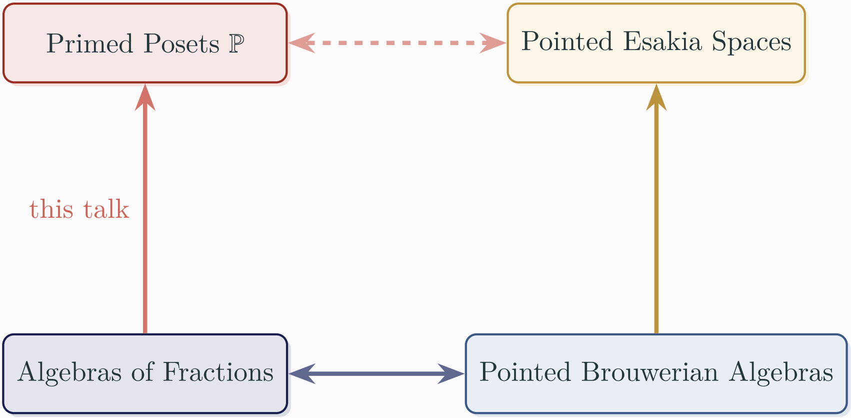
Addition

$$a + b = \big(\mathfrak{p} \to (a \wedge b)\big) \wedge (a \vee b)$$

The Normalizing Map π

$$\pi(a, b) = \langle (a \to b) \to (a \wedge \mathfrak{p}), a \to b \rangle$$

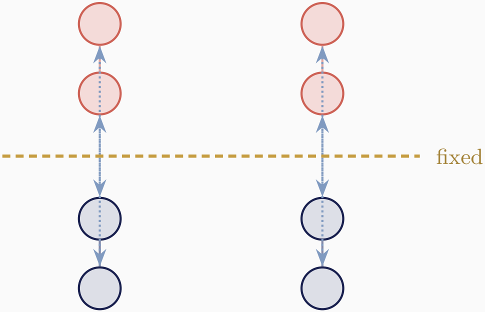
Theorem (Galatos & Přenosil): Categorical Equivalence between these structures.


Primed Posets $\mathbb{P}$
Pointed Esakia Spaces
this talk
Algebras of Fractions
Pointed Brouwerian Algebras

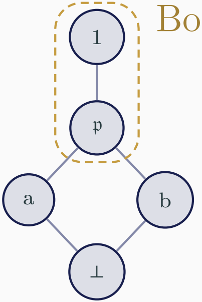
The Semilinear Case

Fussner–Galatos

If $\mathscr{B}$ is semilinear, then $\mathfrak{J}_{\mathscr{B}}$ is a Sugihara monoid and the dual is a reflection: the Esakia dual and its mirror image, glued along the fixed points.



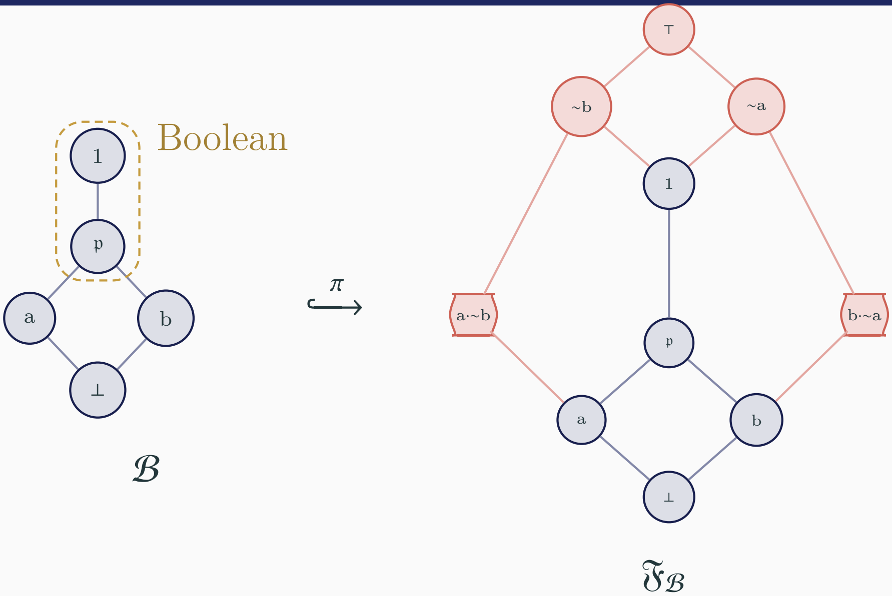
W. Fussner and N. Galatos, (2019).



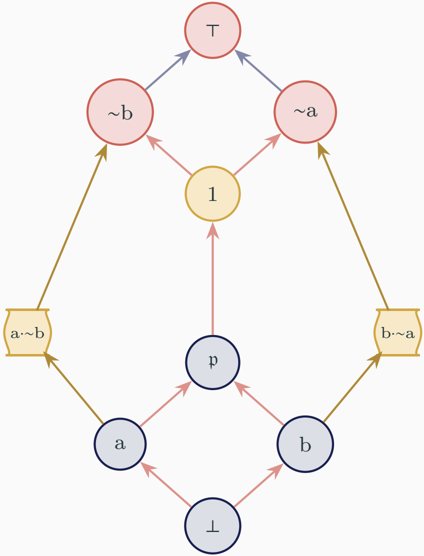
Boolean

$\mathcal{B}$

Building an Algebra of Fractions

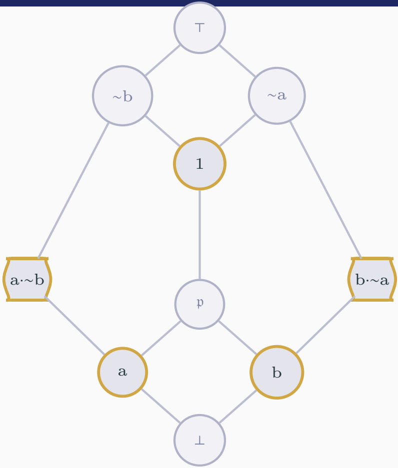


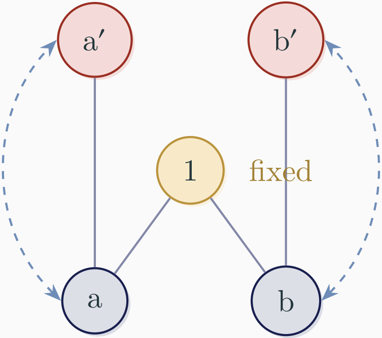
A failure of distributivity



Step 1:
The Shape of an Irreducible

Irreducibles Visually





The prime map $w \mapsto w'$ pairs each new irreducible with its base; 1 is the lone fixed point.

Irreducibles Algebraically

Original Irreducibles

$$\pi\langle \mathfrak{a}, \mathfrak{p} \rangle$$

New Irreducibles

$$\langle \mathfrak{k}(\mathfrak{a}) \to \mathfrak{a}, \mathfrak{k}(\mathfrak{a}) \rangle$$

The ' involution

$$\pi\langle \mathfrak{a}, \mathfrak{p} \rangle \iff \langle \mathfrak{k}(\mathfrak{a}) \to \mathfrak{a}, \mathfrak{k}(\mathfrak{a}) \rangle$$

Ordering Irreducibles

Conicality

A is conical for c if every element of $\downarrow c$ is comparable to some element of A.

Order on the irreducibles

- $\pi\langle a, \mathfrak{p}\rangle \leq \pi\langle b, \mathfrak{p}\rangle \iff a \leq b$
- $\pi\langle a, \mathfrak{p}\rangle \leq \langle k(b) \to b, k(b)\rangle \iff$ b conical for a
- $\langle k(b) \to b, k(b)\rangle \leq \langle k(a) \to a, k(a)\rangle' \iff$ $b \leq a$ and b conical for a

Step 2:
Resolving Non-Distributivity

Problem 1

No topology of clopen upsets.

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Problem 2

How to dualize morphisms with non-prime elements.

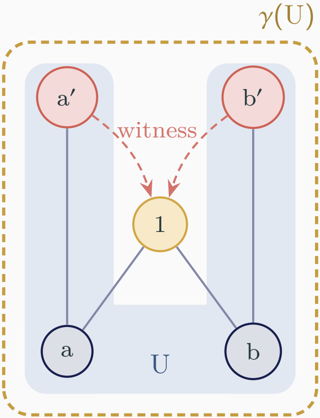
Problem 1: Topology $\rightsquigarrow \gamma$

The closure operator

$$\gamma(\mathrm{U}) = \left\{ \mathrm{w} \;\middle|\; \mathrm{u} \leq \mathrm{w} \Rightarrow \mathrm{u}' \in {\uparrow}\mathrm{U}' \right\}$$

Theorem

$\mathrm{x} \leq \bigvee \mathrm{S} \iff \mathrm{x} \in \gamma({\downarrow}\mathrm{S}).$



Morphism Lemmas

Decomposition Lemma

We can decompose every element of an algebra of fractions:

$$\langle \mathrm{a}, \mathrm{b} \rangle \;=\; \boldsymbol{\pi}\langle \mathrm{a}, \mathfrak{p} \rangle \cdot {\sim}\boldsymbol{\pi}\langle \mathrm{b}, \mathfrak{p} \rangle$$

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Canonical Preservation Lemma

If $h_0 : \mathcal{A} \to \mathcal{B}$ is a Brouwerian homomorphism and $\langle a, b \rangle \in \mathfrak{F}_A$, then $\langle h_0(a), h_0(b) \rangle \in \mathfrak{F}_B$

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Theorem

Every morphism $h : \mathfrak{F}_A \to \mathfrak{F}_B$ can be understood via

$$\langle a, b\rangle \mapsto \langle h_0(a), h_0(b)\rangle$$

where h_0 is the map induced by h and the $\pi\langle -, \mathfrak{p}\rangle$-embedding.

The Dual of a Morphism

Points are irreducible upsets

$$W \;\cong\; \{\uparrow j \mid j \in J(\mathfrak{F}_{\mathcal{B}})\}, \quad \text{ordered by } \supseteq.$$

Theorem

For a homomorphism $h\colon \mathfrak{F}_A \to \mathfrak{F}_B$, taking preimages

$$h^{\partial} := \uparrow v \;\longmapsto\; h^{-1}(\uparrow v)$$

is well defined.

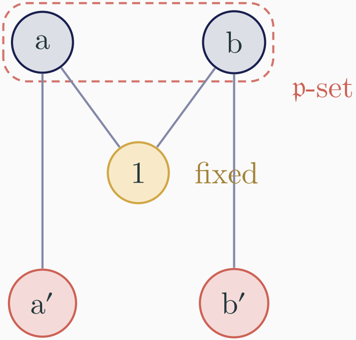
Duality: The Prime Poset Category

Objects of $\mathbb{P}$: Primed Posets

Definition

A poset W with an involution $w \mapsto w'$ such that:

- the 1-set $\{w \mid w \leq w'\}$ is an upset,
- the fixed set $\{w \mid w = w'\}$ is an antichain,
- and obeys the ordering restrictions mentioned earlier.



Definition

Morphisms of $\mathbb{P}$ are functions $\mathrm{f}\colon \mathrm{W}_{\mathcal{B}} \to \mathrm{W}_{\mathcal{A}}$:

1. monotone
2. equivariant: $\mathrm{f}(\mathrm{w}') = \mathrm{f}(\mathrm{w})'$
3. mapping the $\mathfrak{p}$-set into the $\mathfrak{p}$-set
4. a p-morphism on the 1-set.

Esakia Parallel

A morphism of $\mathbb{P}$ is the equivariant extension of an Esakia morphism $\bar{\mathrm{f}}$.

Multiplication, Before and After

In the algebra of fractions

$$\langle \mathrm{p}, \mathrm{q} \rangle \cdot \langle \mathrm{r}, \mathrm{s} \rangle \;=\; \pi\Big(\, \mathrm{p} * \mathrm{r}, \quad \mathrm{q} + \mathrm{s} \,\Big)$$

With $\Phi = (\mathrm{p} \wedge \mathrm{r}) \to \Big[(\mathfrak{p} \to (\mathrm{q} \wedge \mathrm{s})) \wedge (\mathrm{q} \vee \mathrm{s})\Big]$, multiplication is

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On the dual side

$$\mathrm{u} \cdot \mathrm{v} = \begin{cases} \mathrm{v} & \text{if v is new and } \mathrm{b}(\mathrm{v}) \leq \mathrm{b}(\mathrm{u}), \\ \mathrm{u} & \text{if u is new and } \mathrm{b}(\mathrm{u}) \leq \mathrm{b}(\mathrm{v}), \\ \mathrm{u} \wedge \mathrm{v} & \text{otherwise.} \end{cases}$$

Commutative and idempotent by inspection.

Recovering the Algebra

The dual algebra W^{∂} lives on the γ-closed upsets:

Downset Operations

$$U \wedge V = U \cap V \qquad U \vee V = \gamma(U \cup V)$$

$$\sim U = (\uparrow (U'))^{c} \qquad \mathfrak{p} := \text{the } \mathfrak{p}\text{-set}$$

The Duality

Theorem

$(-)^\partial$ is a dual equivalence between finite algebras of fractions and finite primed posets:

$$\begin{tikzcd} \mathrm{Frac} \arrow[r, bend left, "(-)^\partial"] & \mathbb{P} \arrow[l, bend left, "(-)^\partial"] \end{tikzcd}$$

Natural isomorphisms

$$\eta\colon \mathfrak{F}_{\mathcal{B}} \cong \mathfrak{F}_{\mathcal{B}}^{\partial\partial}, \qquad \mathrm{u} \mapsto \{\mathrm{F} \mid \mathrm{u} \in \mathrm{F}\}$$

$$\epsilon\colon \mathrm{W} \cong \mathrm{W}^{\partial\partial}, \qquad \mathrm{w} \mapsto {\uparrow}\mathrm{w}$$

Moving to the Infinite

Thank You