On the structure of involutive partially ordered monoids as models of multiplicative fragments of linear logic

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TACL, July 27 2026

Co-funded by the European Union

Ministry of Education, Youth and Sports of the Czech Republic

[1]This work has been funded by a grant from the Programme Johannes Amos Comenius under the Ministry of Education, Youth and Sports of the Czech Republic, CZ.02.01.01/00/23_025/0008711.

Outline

- **Involutive partially ordered monoids** as models of multiplicative linear logic MLL, non-commutative MLL, cyclic CyMLL, relevant $\mathbf{RW}^t_m$
- Examples of ipo-monoids: binary relations, pointed groups, po-groups, reducts of **Sugihara monoids**, products of a group and Boolean algebra
- Posets underlying residuated po-magmas and po-semigroups
- The structure of commutative **ipo-monoids with a Sugihara or Boolean unital component**

Involutive po-monoids

An **involutive po-monoid** (ipo-monoid) **A** is an algebraic structure $(A, \leq, \cdot, {\sim}, -, 0)$ such that $(A, \leq)$ is a poset, $\cdot$ is **associative** and

$$\text{(lin)} \qquad x \leq y \iff x \cdot {\sim}y \leq 0 \iff -y \cdot x \leq 0$$

Without associativity (lin) entails:

- $-, \sim$ form a dual Galois connection.
- $-, \sim$ are order reversing, $\quad -{\sim}x \leq x, \qquad {\sim}{-}x \leq x,$
- ${\sim}{-}{\sim}x = {\sim}x \quad$ and $\quad -{\sim}{-}x = -x.$
- ${\sim}{-}x = x = -{\sim}x$, i.e., $\sim, -$ are **involutions**.

Using associativity:

- ${\sim}0 = -0$ **is an identity element**. Naturally ${\sim}0$ is denoted by 1.
- $\cdot$ **is residuated**: (res) $xy \leq z \iff x \leq -(y \cdot {\sim}z) \iff y \leq {\sim}(-z \cdot x)$.
  Therefore $z/y = -(y \cdot {\sim}z)$ and $x\backslash z = {\sim}(-z \cdot x)$.
- $x \leq y \implies xz \leq yz$ and $zx \leq zy$.

Algebraic models of multiplicative linear logic (MLL)

The previous results show that ipo-monoids are po-algebraic models of the **non-commutative, non-cyclic multiplicative linear logic** ($x + y = -({\sim}y \cdot {\sim}x) = {\sim}(-y \cdot -x)$).

With cyclicity ($xy \leq 0 \Leftrightarrow yx \leq 0$) added, we obtain models of **cyclic multiplicative linear logic** (CyMLL), where ${\sim}x = -x$.

With commutativity ($xy \leq yx$) added, we obtain models of the **multiplicative linear logic**. MLL entailment has FMP and is decidable (NP-complete!).

Some of these algebras (reducts of distributive commutative involutive residuated lattices) are models of **multiplicative fragment of contraction-free relevance logic** RW_m^t.

ipo-monoids come in a form of **group-indexed unions of directed (bounded) connected components** $\bigcup_{g \in G} A_g$. Depending on additional axioms or assumptions on the unital component A_e, we can say a little bit more.

Some of the results were discovered with the help of Prover9/Mace4.

Examples: binary relations, pointed groups, po-groups

Any set of **binary relations** $A \subseteq \mathcal{P}(X^2)$ closed under composition and **complement-converse** $\sim R = -R = X^2 \setminus R^{-1}$.
Similarly, **weakening relations** on a poset with composition and complement-converse.

A **pointed group** $(A, \cdot, ^{-1}, 1, 0)$ is a group with an (arbitrary) constant 0.
(Let $\leq$ be equality and define $\sim x = x^{-1} \cdot 0$ and $-x = 0 \cdot x^{-1}$).

A **partially ordered group** is a structure $\mathbf{A} = (A, \leq, \cdot, ^{-1}, 1)$ such that

1. $(A, \leq)$ is a poset
2. $(A, \cdot, ^{-1}, 1)$ is a group and
3. multiplication is order preserving (hence $^{-1}$ is order reversing).

$\implies$ $0 = 1^{-1} = 1$ and **cyclicity** holds: $\sim x = -x = x^{-1}$.

Examples: $(\mathbb{Z}, \leq, +, -, 0)$, $(\mathbb{R}, \leq, +, -, 0)$, $(\mathbb{Q}^+, \leq, \cdot, ^{-1}, 1)$, **all** groups (where $\leq$ is $=$, e.g. $(\mathbb{Z}, +, -, 0)$)

Sugihara ipo-monoids (reducts of algebraic models of RM_m)

$\mathbf{S}_{2n}$: a $2n$-element chain $\quad a_n < \cdots < a_2 < \mathbf{0} < \mathbf{1} < b_2 < \cdots < b_n$

$\mathbf{S}_{2n-1}$: a $2n$–1-element chain $\quad a_n < \cdots < a_2 < \mathbf{0} = \mathbf{1} < b_2 < \cdots < b_n$.

Define $\quad a_1 = 0, \qquad b_1 = 1, \qquad \sim a_i = -a_i = b_i, \qquad \sim b_i = -b_i = a_i,$

$$a_i \cdot a_j = a_{\max(i,j)}, \qquad b_i \cdot b_j = b_{\max(i,j)}, \qquad a_i \cdot b_j = b_j \cdot a_i = \begin{cases} b_j & \text{if } i < j \\ a_i & \text{otherwise} \end{cases}$$

$(\mathbf{S}_m, \leq, \cdot, \sim, -, 0)$ is the ***m*-element Sugihara ipo-monoid** (the m-element Sugihara monoid lattice-free reduct)

$\mathbf{S}_2 = \mathbf{2}$ is the two-element **Boolean algebra**

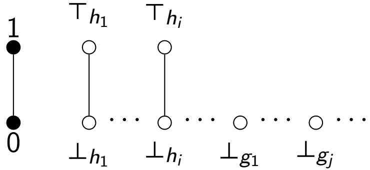
$\mathbf{S}_3$ is also known as the **Gaifman-Pratt ipo-monoid**

Models from products of groups and Boolean algebras

Let **G** be a group with subgroup **H**.

Define $\mathbf{2_{G,H}} = \mathbf{H} \times \mathbf{S}_2 \,\cup\, (\mathbf{G} \setminus \mathbf{H}) \times \mathbf{S}_1$ with

$(g, a) \cdot (g', a') = (gg', aa')$ and $\sim(g, a) = (g^{-1}, \sim a) = -(g, a)$.



Then **A** is an ipo-monoid. **A** is a **conuclear image** of $\mathbf{G} \times \mathbf{2}$ (the conucleus is the map $(g, x) \mapsto (g, \top_g \cdot \top_{g^{-1}} \cdot x)$).

Conuclei on ipo-monoids

A **conucleus** on an ipo-monoid is a **monotone interior** operator:

- $\delta(x) \leq x$
- $\delta(\delta(x)) = \delta(x)$

such that

- $\delta(x) \cdot \delta(y) \leq \delta(xy)$
- $\delta(1) = 1$
- $\delta(-(\delta({\sim}\delta(x)))) \leq \delta(x)$ and $\delta({\sim}(\delta(-\delta(x)))) \leq \delta(x)$

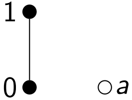
For $x \in \delta[A]$ define ${\sim}_\delta x = \delta({\sim}x)$ and $-_\delta x = \delta(-x)$. Then

$$\mathbf{A}_\delta = (\delta[A], \leq, \cdot, \delta(0), {\sim}_\delta, -_\delta)$$

is an ipo-monoid.

$\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ and $\mathbb{Z}_2 \times \mathbf{2}$

One of the simplest examples of ipo-monoids $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ arises as a conuclear image of the product algebra $\mathbb{Z}_2 \times \mathbf{2}$:



$a = {\sim}a$ acts as an infectious value, but $a^2 = 0$, and $a^3 = a$.

We will later discuss inequational bases for the (po-)varieties generated by $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ and by $\mathbb{Z}_2 \times \mathbf{2}$.

Posets underlying ipo-monoids (rpo-magmas)

In a poset define $x \leftrightsquigarrow y$ if x and y are in the same connected component.

Theorem

Posets underlying ipo-monoids (rpo-magmas) are down-directed and up-directed. Thus in a finite case, ***connected components are bounded****.*

For any partially ordered algebra the relation $\leftrightsquigarrow$ *is a* ***congruence****.*

For a rpo-magma $\mathbf{A}$ *the quotient algebra* $\mathbf{A}/\leftrightsquigarrow$ *is a* ***quasigroup*** *with* $\leq$ *as equality, i.e.,* $xy = z \iff x = z/y \iff y = x\backslash z$.

Conversely, from any quasigroup $\mathbf{Q}$ *and a pairwise disjoint family of* ***bounded*** *posets* A_q *for* $q \in \mathbf{Q}$*, one can construct an rpo-magma with poset* $\bigcup_{q\in\mathbf{Q}} A_q$.

For any rpo-semigroup $\mathbf{A}$ *(and therefore* ***for any ipo-monoid****) the quotient* $\mathbf{A}/\leftrightsquigarrow$ *is a* ***group****.*

Ipo-monoids with additional assumptions

The connected components of **A** are denoted A_g for $g \in \mathbf{A}/\!\rightsquigarrow$, with the **unital component** $1 \in A_e$. If any component is bounded, they are all bounded.

The **order** of $x \in \mathbf{A}$ is the smallest positive n such that $x^n \in A_e$, or ∞ if no finite n exists. If every element has a finite order, they all divide the order of the group, so $\exists n \forall x (x^n \rightsquigarrow 1)$. We are interested in properties of the map $f : x \mapsto x^n$, namely when it is an order embedding and what is its image.

In particular we look at

- Ipo-monoids where $0 \leq 1$, and A_e is idempotent
- Ipo-monoids with $x^{n+1} = x$ for some n (**n-periodic contraction/mingle**)
- weakly integral ipo-monoids ($1 \leq x \Rightarrow 1 = x$)
- Ipo-monoids where A_e is an ipo-**Sugihara monoid**, or a **Boolean algebra**.

What about cyclicity/commutativity?

- Ipo-monoids consisting of **finite chains** are **cyclic**.
- Weakly integral po-monoids with $0 \leq 1$ are **cyclic**, because their components are bounded ($0x = x0$ is the bottom element $\perp \leq x$).
- **Cyclic** ipo-monoids with a **Boolean** A_e are **commutative** if they satisfy $x^{n+1} = x$ for some $n \geq 2$. Not all cyclic ipo-monoids with a Boolean A_e are commutative.
- If (every element of **A** has a finite order), $A_e = \{x \mid x \leq 0 \vee 1 \leq x\}$, $0 \leq 1$ and A_e is idempotent, then
    - A_e is cyclic if and only if **A** is **balanced** ($x \cdot {\sim}x = -x \cdot x =: 0_x$ and $x\backslash x = x/x =: 1_x$)
    - If A_e is cyclic then A_e is a commutative subalgebra of **A**,
    - If A_e is cyclic and A is finite, then **A** is cyclic.
- Ipo-monoids with a **Sugihara** $A_e = S_m$ are all **cyclic** (and therefore balanced).
- There are finite single-component non-cyclic ipo-monoids (6 elements), even chains (7 elements).

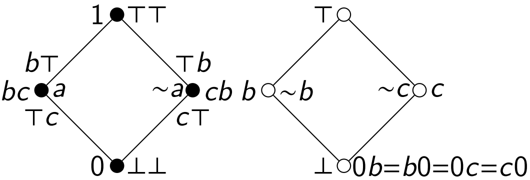


Figure: Noncommutative cyclic ipo-monoid with Boolean unital component. Note that $x^3 = x$ fails in this po-algebra since $b^3 = b \cdot {\sim}b \cdot b = 0b = \bot$.

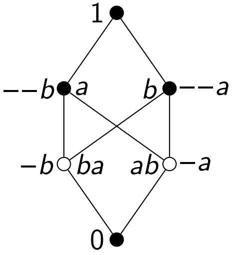


Figure: Smallest noncyclic ipo-monoid with a single connected component.

Commutative ipo-monoids where $0 \leq 1$ and A_e idempotent

Lemma

Let **A** *be a commutative ipo-monoid with a unital component that is idempotent. Then*

- *A_e is a locally-integral ipo-monoid. Positive elements A^+ of A_e form a join-semilattice.*
- **A** *is steady:* $1_x \cdot 1_y = 1_{xy}$

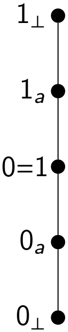
Proof (sketch).

A is commutative, and therefore cyclic and balanced. The units $1_x = x \backslash x \geq 1$ are positive, $1_x \cdot x = x$. If $x \in A_e$, then $x \leq 1_x$ and $x \backslash 1_x = 1_x$ by idempotence.
By Thm. 5.1 of the paper below, **fibers $A_x = \{y \mid 1_x = 1_y\}$ of A_e are Boolean**.
Proof of steadiness essentially uses commutativity. □

A_e can be constructed as a **Płonka sum** of its fibers A_x, indexed by the join semilattice A^+. **Which ipo-monoids A can we construct using Płonka sums?**

J. Gil-Férez, P. Jipsen, M. Sugimoto: Locally integral involutive PO-semigroups, Fundamenta Informaticae, Volume 195, 1-4, 7, (2025) 1–29

Example: $\mathbf{S}_5$



$$\mathbf{S}_5 = \mathbf{1} \uplus_{\varphi_1} \mathbf{2} \uplus_{\varphi_2} \mathbf{2}$$

Construction of Płonka sum of two ipo-monoids

Let $\mathbf{A}, \mathbf{B}$ be disjoint involutive residuated lattices or ipo-monoids, and $\varphi : \mathbf{A} \to \mathbf{B}$ a monoid homomorphism such that φ preserves existing joins (hence preserves the order) and $\varphi(x) \neq 0^{\mathbf{B}}$ for all $x \in A$.

On $A \cup B$, define $\sim\ = \ \sim^{\mathbf{A}} \cup \sim^{\mathbf{B}}$, $- = -^{\mathbf{A}} \cup -^{\mathbf{B}}$,

$$x \cdot y = \begin{cases} x \cdot^{\mathbf{A}} y & \text{if } x, y \in A, \\ x \cdot^{\mathbf{B}} y & \text{if } x, y \in B, \\ \varphi(x) \cdot^{\mathbf{B}} y & \text{if } x \in A \text{ and } y \in B, \\ x \cdot^{\mathbf{B}} \varphi(y) & \text{if } x \in B \text{ and } y \in A, \end{cases} \qquad x \leq y \iff x \cdot {\sim} y = 0^{\mathbf{A}}$$

and let $\mathbf{A} \uplus_{\varphi} \mathbf{B} = (A \cup B, \leq, \cdot, \sim, -, 0^{\mathbf{A}})$ be the **binary Płonka sum**.

Theorem

For any ipo-monoids $\mathbf{A}, \mathbf{B}$ the po-algebra $\mathbf{A} \uplus_{\varphi} \mathbf{B}$ is an ipo-monoid.

Decomposition into Płonka sum of two ipo-monoid

The term 1_x is defined as $-(x \cdot {\sim}x) = x/x$.

An ipo-monoid is **balanced** if $x/x = x\backslash x$ and **steady** if $1_x \cdot 1_y = 1_{xy}$.

Theorem

Let $\mathbf{C}$ *be a balanced steady involutive residuated lattice or ipo-monoid and assume* (p, q) *is a splitting pair of* ***central idempotent*** *elements in* $\mathbf{C}^+$*, i.e.,* $\mathbf{C}^+ = {\downarrow}p \uplus {\uparrow}q$.

Let $A = \{x \in C \mid 1_x \leq p\}$, $B = \{x \in C \mid q \leq 1_x\}$ *with restricted operations from* $\mathbf{C}$.

Then $\mathbf{A}$ *is a subalgebra,* $\mathbf{B}$ *is a 1-free subalgebra of* $\mathbf{C}$ *with identity* q *and* $C = A \uplus B$.

Define $\varphi : A \to B$ *by* $\varphi(x) = x \cdot q$.

Then φ *is a monoid homomorphism that preserves all existing joins and* $\mathbf{C} = \mathbf{A} \uplus_\varphi \mathbf{B}$.

Ipo-monoids with $x^{n+1} = x$, and $0 \le 1$

n-periodic contraction/mingle $x^{n+1} = x$ (holds for commutative ipo-monoids of finite order with Sugihara or Boolean A_e).

- $x^{n+1} = x$ means $x \sqsubseteq x^n$ (the monoid-induced order)
- $x^{n+1} = x$ implies $x^n \leftrightsquigarrow 1$, and x^n **is idempotent**, and $(x^n)^n = x^n$.
- $x^{n+1} = x$ IFF $x \cdot {\sim}x \le x^n \le x \backslash x$ and $-x \cdot x \le x^n \le x/x$

Lemma

In an ipo-monoid A satisfying $x^{n+1} = x$ for some n:

- *If $x, y \in A_g$, then $f : A_g \to A_e$ given by $f(x) = x^n$ is an* ***order embedding****.*
- *Fixpoints of f coincide with the idempotents of A.*

If moreover $0 \le 1$,

- *fixpoints of f form a cyclic balanced subalgebra of A (of A_e),*
- *$A' = \{x \mid x \le 1 \text{ or } 1 \le x\}$ is a subalgebra of $Idmp(A)$, where $\{x \mid 0 \le x \le 1\}$ is a Boolean algebra, and $\{x \mid x \le 0 \text{ or } 1 \le x\}$ is a (reduct of) Sugihara monoid.*

Commutative ipo-monoids with Boolean unital component

Lemma

Let **A** *be a commutative ipo-monoid with a unital component that is a Boolean algebra* **B** *and denote the group* $\mathbf{A}/\!\rightsquigarrow$ *by* **G**. *Then* **A** *is a conuclear image of* $\mathbf{G} \times \mathbf{B}$ *and all components of* **A** *are isomorphic to principal ideals of* **B**.

Proof (sketch).

The conuclear map on $\mathbf{G} \times \mathbf{B}$ is defined by $\delta((g, x)) = (g, \top_g \cdot \top_{g^{-1}} \cdot x)$. The proof uses that $\top_g \cdot \top_{g^{-1}} \cdot \top_g = \top_g$. $\square$

In case every element of **A** has a finite order, we obtain the following insight: let order of $x, y \in A_g$ be n. Then

- $x^n = y^n$ iff $x = y$, and
- $x = x^{n+1} = x \cdot \top_g \cdot \top_{g^{-1}}$.

We will use **conuclear images of $\mathbf{G} \times \mathbf{B}$** as **fibers** of **Płonka sums** to construct commutative ipo-monoids (e.g. those with Sugihara unital component).

Structure of ipo-monoids with $A_e \cong \mathbf{S}_k$

Theorem

Let $\mathbf{A}$ *be an ipo-monoid with* $A_e \cong S_k$ *where every element is of a finite order. Then* $\mathbf{A}$ *is a disjoint union of chains, each with* $\leq k$ *elements, and* $\cdot$ *determined by subgroups of* $\mathbf{A}/\!\!\rightsquigarrow$*. Moreover,* $\mathbf{A}$ *is cyclic and balanced.*

Proof (outline).

Let $\mathbf{G} = \mathbf{A}/\!\!\rightsquigarrow$. The map $f : A_g \to A_e$ mapping $x \mapsto x^{o(x)}$ is an **order embedding.** Therefore all connected components are chains with $\leq k$ elements.

For $1 \leq l \leq k$, let $\mathbf{H}_l = \{g \in \mathbf{G} \mid l \leq |A_g|\}$.

Then $\mathbf{H}_k \leq \cdots \leq \mathbf{H}_2 \leq \mathbf{H}_1 = \mathbf{G}$ is a nested sequence of subgroup of $\mathbf{G}$ and these subgroups uniquely determine the multiplication on the algebra.
Every element of $\mathbf{A}$ is either a **lower element** ($0_x = x^n \leq 0$) or an **upper element** ($1 \leq x^n = 1_x$), therefore $\mathbf{A}$ is balanced. As all the components are finite chains, then $\mathbf{A}$ is cyclic. $\square$

We can obtain this insight without assuming finite order, if A is balanced.

Structure of ipo-monoids with $A_e \cong \mathbf{S}_k$

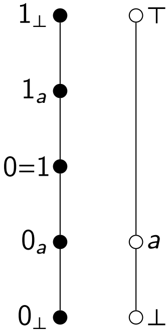


Figure: The ipo-monoid $\mathbf{1} \uplus_{\varphi_1} \mathbf{2}_{\mathbb{Z}_2,\{e\}} \uplus_{\varphi_2} (\mathbb{Z}_2 \times \mathbf{2})$ with S_5 as unital component

Structure of commutative ipo-monoids with $A_e \cong \mathbf{S}_k$ - Płonka sums

Every **commutative** ipo-monoid $\mathbf{A}$ with $A_e \cong \mathbf{S}_k$ can also be constructed using a **Płonka sum** over

- a linear join-semilattice of positive elements A^+ with the bottom element 1,
- A^+ directed system of ipo-monoids $A_x = \{y \mid 1_y = 1_x\}$ (fibers)
- $|A_x \cap A_g| \leq 2$
- fibers are ipo-monoids of the form $\mathbf{2}_{\mathbf{G},\mathbf{H}}$.

Every commutative ipo-monoid A with $A_e \cong \mathbf{S}_k$ and $\mathbf{G} = \mathbf{A}/\!\!\leftrightsquigarrow$ is a conuclear image of $\mathbf{G} \times \mathbf{S}_k$.

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Structure of commutative ipo-monoids with $A_e \cong \mathbf{S}_k$ - Płonka sums

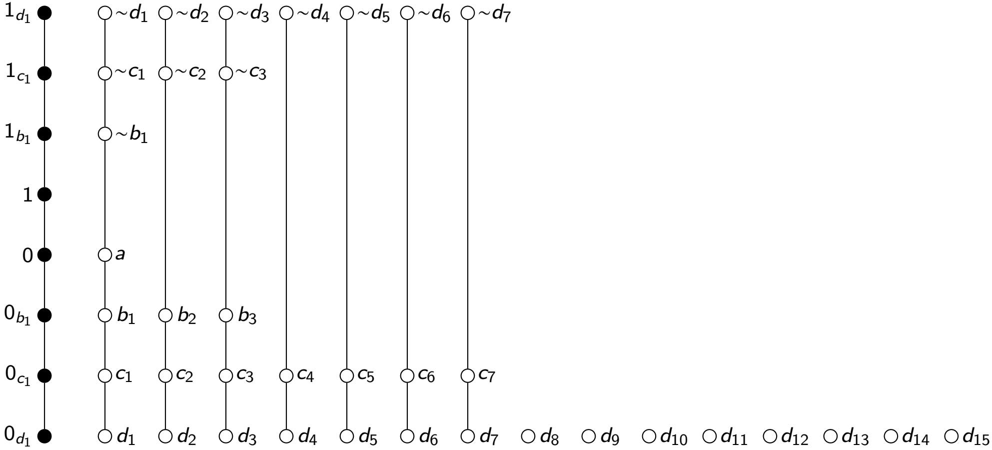
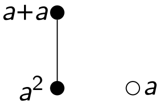


Figure: Example of a Płonka sum $\mathbf{2}_{\mathbb{Z}_2,\{e\}} \uplus_{\varphi_1} \mathbf{2}_{\mathbb{Z}_2^2,H_1} \uplus_{\varphi_2} \mathbf{2}_{\mathbb{Z}_2^3,H_2} \uplus_{\varphi_3} \mathbf{2}_{\mathbb{Z}_2^4,H_3}$

$\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ and $\mathbb{Z}_2 \times \mathbf{2}$-generated (po-)varieties



The order is definable: $x \leq y$ **IFF** $1 = x\backslash y$.
The term $\tau(x) := {\sim}x^2 \cdot (x + x)$ equals 1 if $x = a$, and equals 0 otherwise.
(quasi-)inequational bases (CONJECTURE) for the varieties generated by

- $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$
  - Boolean axioms instances in $(\cdot, +, \sim, 0)$ for x^2
  - ipo-monoid axioms
  - $x^3 = x$
  - $\tau(x) \cdot \tau(y) \leq \tau(x) \cdot ({\sim}(xy))^2$
- $\mathbb{Z}_2 \times \mathbf{2}$
  - Boolean axioms instances in $(\cdot, +, \sim, 0)$ for x^2
  - ipo-monoid axioms
  - $x^3 = x$
  - ${\sim}(x^2) = ({\sim}x)^2$

Thank you!

Co-funded by the European Union

Ministry of Education, Youth and Sports of the Czech Republic

This work has been funded by a grant from the Programme Johannes Amos Comenius under the Ministry of Education, Youth and Sports of the Czech Republic, CZ.02.01.01/00/23_025/0008711.

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