Small Dialectica Categories

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Joint work with Colin Bloomfield and Peter Jipsen

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"Dialectica categories are not hard. Let me show you the smallest ones."

Thank you!

 

http://arxiv.org/abs/2607.23430

Thanks Nick and Nick, as well as all organizers!

 

2 / 41

One construction, many logics

Can one mathematical construction generate models for many different logics?

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Dialectica categories do exactly that.
Turn a couple of dials and out comes a model of

- the Lambek calculus
- Intuitionistic (Linear) Logic
- Classical Linear Logic
- and plenty in between

Same blueprint. Different logics. No extra effort.

Where did this machine come from?

1942/1958 — Gödel
The Dialectica interpretation turned proofs into computational witnesses. Witnesses and counter-witnesses were already playing dual roles — the idea was there from the start, waiting for a category to live in.

1987/1990 — de Paiva
Swap computable functionals for morphisms in an arbitrary category. The proof interpretation becomes a category construction. Or two.

Proof interpretation $\Longrightarrow$ Category constructions.

A family of Dialectica models

Three independent dials, one family of models:

$$\mathbf{D}_L(\mathbf{C}) \qquad \mathbf{G}_L(\mathbf{C})$$

- **Construction, D or G** — determines the morphism structure:
  - **D**, Gödel-style — the backward map may depend on the forward witness. Models *intuitionistic* linear logic.
  - **G**, Girard-style — the backward map does not. Models *classical* linear logic.
- **Lineale,** L — determines the truth values, or weights (a lineale is just a symmetric monoidal closed poset)
- **Base category, C** — determines the categorical structure

Turn any of the three, and you land in a different logic. It's a whole family — today we're only visiting one branch of it.

Today's corner of the family

Today we make the smallest possible choices:

$$L = 2, \qquad \mathbf{C} = \mathbf{2}$$

and compare the two constructions

$$\mathbf{D}_2(\mathbf{2}) \quad \text{vs.} \quad \mathbf{G}_2(\mathbf{2}).$$

Small on purpose. What happens when Dialectica has nowhere left to shrink?

Why study the smallest examples?

Small examples often reveal the essential structure.

- We understand groups by studying small groups.
- We understand posets by studying finite posets.
- We understand categories by studying walking objects.

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So today we ask the obvious next question:

What happens in the smallest Dialectica categories?

Today's story

We will answer three questions.

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2. How much of the general theory already shows up in these tiny examples?
3. What do they reveal about the modality !...

Three small questions, a couple of hours, no prerequisites beyond a pencil.

The smallest base category

We start with the smallest non-trivial base category, the “walking arrow”:

$$\mathbf{C} = \mathbf{2} = (0 \longrightarrow 1)$$

Two objects. Three morphisms. That’s the entire setting.

Pair it with the Boolean lineale

$$L = 2 = \{0 < 1\},$$

and you get the smallest possible instances of the Dialectica constructions $\mathbf{D}_2(\mathbf{2})$ and $\mathbf{G}_2(\mathbf{2})$— as small as this story gets.

Objects

An object of both constructions is a triple

$$(u, x, \alpha), \qquad \alpha \rightarrowtail u \times x, \qquad u, x, \alpha \in \{0, 1\}.$$

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In a poset like $\mathbf{2}$ that's just the inequality $\alpha \leq u \wedge x$.

Three independent bits suggest $2^3 = 8$ triples.

But α isn't a free third bit — it's a **subobject** of $u \times x$, so $\alpha \leq u \times x$.

Hence an object like $(0, 0, 1)$ doesn't make sense, as 1 is not smaller than 0×0.

Let's meet the five that can be objects.

The five objects

Every object of $\mathbf{D}_2(\mathbf{2})$ and $\mathbf{G}_2(\mathbf{2})$ is isomorphic to exactly one of these five:

a	$\top$	z	$\bot$	i
$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(1,1,1)$

Five objects are all we need. The entire category, computed by hand, fits on one line.

The five objects, and one that shouldn't be there

A linear logician expects exactly four nullary connectives:

$\top$	z	i	$\bot$
additive true	additive false	multiplicative true	multiplicative false

That's four. We found five.

The extra one, $(0, 0, 0)$, doesn't correspond to any logical constant at all. We call it the **odd** one — and it turns out to be worth keeping.

Same objects, different morphisms

The two constructions have exactly the same objects. They differ only in the definition of morphism.

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The backward map in **D** may depend on the forward witness u.

One dependency. That's the entire difference between the two constructions — and it's enough to reshape everything downstream.

Two notions of morphism

Girard construction $\mathbf{G}_L(\mathbf{C})$

$$\begin{tikzcd} u \arrow[d, "f"'] & x \arrow[l, "\alpha"', "|" marking] \\ v & y \arrow[l, "\beta", "|" marking] \arrow[u, "F"'] \end{tikzcd}$$

$$f : u \to v, \qquad F : y \to x$$

$$\alpha(u, F(y)) \implies \beta(f(u), y)$$

The counter-witness depends only on the challenge.

Dialectica construction $\mathbf{D}_L(\mathbf{C})$

$$\begin{tikzcd} u \arrow[d, "f"'] & x \arrow[l, "\alpha"', "|" marking] \\ v & y \arrow[l, "\beta", "|" marking] \arrow[u, "F"'] \end{tikzcd}$$

$$f : u \to v, \qquad F : u \times y \to x$$

$$\alpha(u, F(u, y)) \implies \beta(f(u), y)$$

The counter-witness may depend on the chosen witness.

That's the categorical picture. Now take it downstairs.

The same picture, in a poset

Let the base $\mathbf{C}$ be a poset — a Heyting algebra H, or just $\mathbf{2}$. Then every hom-set has at most one element, and *all* the data collapses into inequalities.

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A morphism $(u, x, \alpha) \to (v, y, \beta)$ exists exactly when:

Girard	$u \leq v$	$y \leq x$	$\alpha \wedge (u \wedge y) \leq \beta \wedge (u \wedge y)$
Dialectica	$u \leq v$	$u \wedge y \leq x$	$\alpha \wedge (u \wedge y) \leq \beta \wedge (u \wedge y)$

Two of the three are shared. $F : y \to x$ became $F : u \times y \to x$, and $u \times y$ is just $u \wedge y$.

One substitution, in one inequality. That is the whole difference.

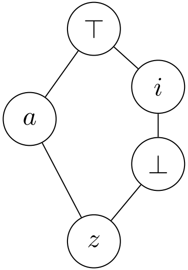
What that buys you, in $\mathbf{G}(\mathbf{2})$

Run the three inequalities on the five objects.
Two maximal chains, sharing top and bottom:

$$z < a < \top \qquad \text{and} \qquad z < \bot < i < \top$$

with a and $\bot$ **incomparable** — $a \to \bot$ needs $y \le x$, i.e. $1 \le 0$, and $\bot \to a$ needs $u \le v$, i.e. $1 \le 0$.

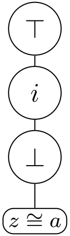
A non-distributive pentagon. Five objects, and already a shape too twisted to be a Heyting algebra.



What that buys you, in $\mathbf{D}(\mathbf{2})$

Same five objects, same three inequalities — except the middle one is now $u \wedge y \leq x$.

$a \to z$**:** in $\mathbf{G}$ this needed $y \leq x$, that is $1 \leq 0$. Blocked.
In $\mathbf{D}$ it only needs $u \wedge y \leq x$, that is $0 \wedge 1 = 0 \leq 0$. Fine.



And $z \to a$ held already. **So** $z \cong a$, and the pentagon collapses to a four-element chain:

$$z \cong a \;<\; \perp \;<\; i \;<\; \top$$

The odd object survives in $\mathbf{G}$ and dies in $\mathbf{D}$. One meet did that.

Why the third inequality is shared

Two of those three inequalities were shared. Here is why: a morphism is the *dashed arrow*, with α and β both pulled back to $\mathrm{Sub}(u \times y)$.

$$\begin{tikzcd}
& A' \arrow[r] \arrow[d, "\langle\pi_1,F\rangle^{-1}(\alpha)"] \arrow[dl, dashed, bend right] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "\alpha"] \\
B' \arrow[r, tail, "(f\times y)^{-1}(\beta)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & u \times y \arrow[r, "\langle\pi_1,F\rangle"] \arrow[d, "f\times y"] & u \times x \\
B \arrow[r, tail, "\beta"'] & v \times y &
\end{tikzcd}$$

18 / 41

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B \arrow[r, tail, "\beta"'] & v \times y &
\end{tikzcd}
$$

Dialectica	the middle row is $\langle \pi_1, F\rangle$,	$F : u \times y \to x$
Girard	it is $u \times F$ instead,	$F : y \to x$

Everything else in the diagram is identical.

Why the third inequality is shared

Downstairs. Every object becomes an element, every map an inequality, every pullback a *meet*. In particular $u \times y$ is $u \wedge y$, and

$$A' = \alpha \wedge (u \wedge y), \qquad B' = \beta \wedge (u \wedge y)$$

The two candidate maps, $\langle \pi_1, F \rangle$ and $u \times F$, both have domain $u \wedge y$ — and in a poset there is at most one arrow, so they are the same map. The dashed arrow exists precisely when

$$\alpha \wedge (u \wedge y) \leq \beta \wedge (u \wedge y)$$

The maps differ upstairs. Downstairs the pullbacks do not.

Which is why the pentagon and the chain part company on $u \wedge y \leq x$ alone.

The additive constants

Straight from the inequalities:

$$(u, x, \alpha) \longrightarrow \top = (1, 0, 0) \quad \text{for every object}$$

$$z = (0, 1, 0) \longrightarrow (u, x, \alpha) \quad \text{for every object}$$

So $\top$ is terminal and z is initial — no extra proof required, they fall straight out.

That leaves the objects $(1, 1, \alpha)$: the unit of the tensor, and its negation. Which means we need the tensor.

The tensor, categorically

Defined once, in an arbitrary base — with products, and for the Girard construction with function spaces too.

Dialectica: witnesses pair with witnesses, challenges with challenges, 'relations' v.

$$(u, x, \alpha) \otimes_D (v, y, \beta) \;=\; \big(u \times v,\;\; x \times y,\;\; `\alpha \otimes \beta'\big)$$

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Girard: the challenge space works harder.

$$(u, x, \alpha) \otimes_G (v, y, \beta) \; = \; \big(u \times v, \;\; x^v \times y^u, \;\; \alpha \otimes \beta\big)$$

A counter-witness is now a pair of maps, $v \to x$ and $u \to y$: each side's forward witness is fed in to produce a counter-witness for the *other* side.

Unit $i = (1, 1, 1)$ in both.

The tensor, in a poset

Dialectica: nothing to translate. Products become meets, and meets are all we need.

$$(u, x, \alpha) \otimes_D (v, y, \beta) \;=\; \big(u \wedge v,\;\; x \wedge y,\;\; \alpha \wedge \beta\big)$$

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Girard: the function spaces become Heyting implications, $x^v = v \to x$.

$$(u, x, \alpha) \otimes_G (v, y, \beta) \;=\; \big(u \wedge v,\;\; (v \to x) \wedge (u \to y),\;\; \alpha \wedge \beta\big)$$

Meets alone are not enough: forming x^v needs the base *cartesian* closed — a Heyting algebra, not just a meet-semilattice.

Both still have unit $i = (1, 1, 1)$. Next slide: these two lines, run on five objects.

Tensor products, computed

$\mathbf{G}_2(\mathbf{2})$

$\otimes_G$	z	a	$\bot$	i	$\top$
z	z	z	z	z	z
a	z	z	a	a	a
$\bot$	z	a	$\bot$	$\bot$	$\top$
i	z	a	$\bot$	i	$\top$
$\top$	z	a	$\top$	$\top$	$\top$

$\mathbf{D}_2(\mathbf{2})$

$\otimes_D$	z	$\bot$	i	$\top$
z	z	z	z	z
$\bot$	z	$\bot$	$\bot$	$\top$
i	z	$\bot$	i	$\top$
$\top$	z	$\top$	$\top$	$\top$

Row i is the identity in both — which is why it's called i. The name was earned, not assigned.

The internal hom, categorically

Residuation pins it down:

$$\alpha \otimes \beta \leq \gamma \iff \alpha \leq \beta \multimap \gamma.$$

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Girard:

$$(u, x, \alpha) \multimap_G (v, y, \beta) \; = \; \big(v^u \times x^y,\; u \times y,\; h\big)$$

$$h\big((f_1, f_2), (u', y')\big) \iff \alpha(u', f_2(y')) \Rightarrow \beta(f_1(u'), y')$$

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Dialectica: same shape, same u-dependency as always — now f_2 gets to see u':

$$(u, x, \alpha) \multimap_D (v, y, \beta) \;=\; \big(v^u \times x^{u \times y},\; u \times y,\; h\big)$$

$$h\big((f_1, f_2), (u', y')\big) \iff \alpha(u', f_2(u', y')) \Rightarrow \beta(f_1(u'), y')$$

The internal hom, in a poset

Write w for the first coordinate and p for the second. Then h is forced: the residual, cut down to $w \wedge p$.

Girard:

$$(u, x, \alpha) \multimap_G (v, y, \beta) \;=\; \Big(\underbrace{(u \to v) \wedge (y \to x)}_{w},\; \underbrace{u \wedge y}_{p},\; w \wedge p \wedge (\alpha \to \beta)\Big)$$

Dialectica:

$$(u, x, \alpha) \multimap_D (v, y, \beta) \;=\; \Big(\underbrace{(u \to v) \wedge \big((u \wedge y) \to x\big)}_{w},\; \underbrace{u \wedge y}_{p},\; w \wedge p \wedge (\alpha \to \beta)\Big)$$

Same substitution as before: $u \wedge y$ replaces y.

Cartesian closed base, monoidal closed result — this $\multimap$ is right adjoint to $\otimes$, not to $\wedge$.

Internal hom, computed

$\mathbf{G}_2(\mathbf{2})$

$\multimap_G$	z	a	$\bot$	i	$\top$
z	$\top$	$\top$	$\top$	$\top$	$\top$
a	a	$\top$	a	a	$\top$
$\bot$	z	a	i	i	$\top$
i	z	a	$\bot$	i	$\top$
$\top$	z	a	z	z	$\top$

$\mathbf{D}_2(\mathbf{2})$

$\multimap_D$	z	$\bot$	i	$\top$
z	$\top$	$\top$	$\top$	$\top$
$\bot$	z	i	i	$\top$
i	z	$\bot$	i	$\top$
$\top$	z	z	z	$\top$

Row i is the identity again: $i \multimap w = w$. Residuation, verified by hand, on twenty-five entries.

Linear negation, in $\mathbf{G}(\mathbf{2})$

Define $\neg w = w \multimap_G \bot$ — read off column $\bot$ from the table above:

w	z	a	$\bot$	i	$\top$
$\neg w$	$\top$	a	i	$\bot$	z

- Involutive: $\neg\neg w = w$
- a is self-dual — the odd object negates to itself
- $z \leftrightarrow \top$ swap, $\bot \leftrightarrow i$ swap

Five objects, and negation is already a genuine involution — nothing was assumed, it fell out of the formula.

What we just built has a name

A **lineale** is a poset with a symmetric monoidal closed structure satisfying

$$\alpha \circ \beta \leq \gamma \iff \alpha \leq \beta \multimap \gamma.$$

It is precisely what poset-reflecting a symmetric monoidal closed category gives you — which is why every formula on the last six slides turned into an inequality.

A **cartesian** lineale — where $\circ$ is idempotent — turns out to be just a Brouwerian semilattice, with $\multimap$ collapsing to ordinary Heyting implication.

The walking arrow $\mathbf{2} = (0 \to 1)$, as a poset, is itself the simplest lineale there is.
We have been standing on one since slide 1.

The line that justifies the whole talk

First, tidy up. Objects of $\mathbf{D}(H)$ satisfy $(u, x, \alpha) \cong (u,\, u \wedge x,\, \alpha)$ — the morphism condition only ever sees $u \wedge x$. So let $\mathbf{D}'(H)$ keep the canonical representatives: chains $\alpha \leq x \leq u$ in H, one per object.

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Theorem (Bloomfield) Assuming Choice:

$$\mathcal{P}\,\mathbf{D}_2(\mathbf{Set}) \ \cong\ \mathcal{P}\,\mathbf{D}_2(\mathbf{2}) \ =\ \mathbf{D}'(\mathbf{2})$$

The poset reflection of **all** of $\mathbf{D}(\mathbf{Set})$ — every set, every relation, no size limit — is our four-element chain.

The small model is not a sketch of the theory. Up to poset reflection, it **is** the theory.

Bloomfield, Jipsen, de Paiva. *Dialectica Categories over Heyting Algebras.* arXiv preprint, submitted.

Teaching a category to share, in $\mathbf{D}$

$!$ in $\mathbf{D}(\mathbf{C})$ has a clean description:

$$!(u, x, \alpha) = (u,\ x^*,\ !\alpha)$$

where x^* is the free commutative monoid on x.

In plain English: use the counter-witness x as many times as you like — including never. That is "of course," in full.

One comonad. Cofree, and satisfying the storage axioms on the nose.

$\mathbf{D}$ is the co-Kleisli category of $\mathbf{G}$

In $\mathbf{G}(\mathbf{C})$ the second coordinate varies the wrong way for a free monoid. The fix is a comonad in front — the **reader** $T(u, x, \alpha) = (u,\ x^u,\ \alpha^u)$.

A $\mathbf{G}$-morphism $T(u, x, \alpha) \to (v, y, \beta)$ is a pair $f : u \to v$ and $F : y \to x^u$ — which is to say $F : u \times y \to x$.

That is a $\mathbf{D}$-morphism. $\qquad \mathbf{D}(\mathbf{C}) \;\simeq\; \mathrm{Kl}_T\big(\mathbf{G}(\mathbf{C})\big)$

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Downstairs the exponential is $x^u = u \to x$, and currying is residuation:

$$y \le (u \to x) \iff u \wedge y \le x$$

The one substituted meet that separated the pentagon from the chain is the Kleisli construction, written out.

$\mathbf{D}$ is the co-Kleisli category of $\mathbf{G}$

On the pentagon, $T(u, x, \alpha) = (u,\ u \to x,\ \alpha)$ acts by

$$T(a) = z, \qquad T \text{ fixes } z,\ \perp,\ i,\ \top.$$

An interior operator whose image is exactly the four-element chain — the odd object folded onto z, which is the one identification $\mathbf{D}$ makes and $\mathbf{G}$ refuses.

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And the modalities come along. Upstairs $! = S \circ T$ in $\mathbf{G}$: the free monoid S, after the reader T. Downstairs S has nothing to build on — but T survives, and the two modalities of $\mathbf{D}(H)$ we are about to build live on its co-Kleisli side.

$\mathbf{G}$ was never missing a modality. It was missing one composition step.

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$$i \,\&\, A \,\&\, (A \otimes A) \,\&\, (A \otimes A \otimes A) \,\&\, \ldots$$

But in $\mathbf{D}(H)$ the tensor is a coordinatewise meet, so it is *idempotent* — contraction holds, and the whole series collapses to $i \,\&\, A$:

$$!(u, x, \alpha) \;=\; i \,\&\, (u, x, \alpha) \;=\; \big(u,\; u,\; (x \to \alpha) \wedge u\big)$$

An interior operator with $!\alpha \otimes !\beta \leq !(\alpha \otimes \beta)$; $\mathbf{D}(H)$ models ILL with "of course".

Bloomfield, Jipsen, de Paiva, *Dialectica Categories over Heyting Algebras*, Prop. 8 and Thm. 9.

An adjoint the general construction seems to lack

The 1990 thesis embeds $\mathbf{C}$ into $\mathbf{D}(\mathbf{C})$ by sending u to $(u, u, \text{identity relation})$, and remarks that this *"does not seem to have an easy adjoint."*

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Over a Heyting algebra it does. With $\iota(w) = (w, w, w)$:

$$\iota \dashv f, \qquad f(u, x, \alpha) = u \wedge (x \to \alpha), \qquad !_\iota(u, x, \alpha) = (t, t, t), \;\; t = u \wedge (x \to \alpha)$$

ι is strong monoidal, so $\iota \dashv f$ is a linear/non-linear model and $!_\iota$ a linear exponential comonad.

An adjoint the general construction seems to lack

The 1990 thesis embeds $\mathbf{C}$ into $\mathbf{D}(\mathbf{C})$ by sending u to $(u, u, \text{identity relation})$, and remarks that this *"does not seem to have an easy adjoint."*

Over a Heyting algebra it does. With $\iota(w) = (w, w, w)$:

$$\iota \dashv f, \qquad f(u, x, \alpha) = u \wedge (x \to \alpha), \qquad !_\iota(u, x, \alpha) = (t, t, t), \;\; t = u \wedge (x \to \alpha)$$

ι is strong monoidal, so $\iota \dashv f$ is a linear/non-linear model and $!_\iota$ a linear exponential comonad.

It differs from the previous !: on $\bot = (1, 1, 0)$, $\;!_\iota \bot = a\;$ while $\;!\bot = \bot$.

The poset supplies, for free, an adjoint the general setting was missing.

de Paiva, *The Dialectica Categories*, 1991, §1.4 — Bloomfield, Jipsen, de Paiva, Prop. 7.

From categorical to algebraic LNL models

The term on the last slide, made precise.

Benton's LNL models: a symmetric monoidal adjunction $F \dashv G : \mathcal{L} \to \mathcal{C}$, with ! given by GF.

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Poset-reflect throughout: an **algebraic LNL model** is a monoidal adjunction $f \dashv g : L \to C$ from a lineale L to a cartesian lineale C, with $! = g \circ f$.

And we have one already: $\iota \dashv f$ on $\mathbf{D}(H)$ is exactly this, with $\mathbf{D}(H)$ as L and H as C. For $H = \mathbf{2}$, both sides fit on a napkin.

A linear exponential comonad, in Benton's sense, out of a four-element chain.

Benton 1995 — Maietti, Maneggia, de Paiva, Ritter 2005.

Back to the dials

$\mathbf{D}_L(\mathbf{C})$ has two independent choices. Today we fixed both at the smallest option and turned the D/G switch once.

L — **truth values:** $\mathbf{2}$ yes/no · $\mathbf{3}$ Kleene · $\mathbb{N}$ multiplicities · Ω distances · general lineale weights

$\mathbf{C}$ — **base:** $\mathbf{2}$ walking arrow (today) · H Heyting algebra · **Set** · **Cat**

One click on either dial, and a whole afternoon of new small categories opens up.

The grid, made concrete

$L \backslash \mathbf{C}$	$\mathbf{2}$	H	$\mathbf{Set}$
$\mathbf{2}$	✓ *today*	✓ *preprint*	DC/GC (1989)
$\mathbf{3}$	G only, partial	open	open
$\mathbb{N}$	open	open	$M_{\mathbb{N}}(\mathbf{Set})$ (1991)
Ω	open	open	metric spaces (1973)
L (lineale)	future	future	enriched DC (partial)

Two cells claimed. The rest is blank. The Ω row alone connects straight to Lawvere 1973 — a paper most of this room already knows.

The invitation

Pick your own L and $\mathbf{C}$, write down the three inequalities, and see what logic falls out.

- $\mathbf{D}_3(\mathbf{2})$ — Kleene truth values over the walking arrow: small, finite, unclaimed
- $D_{\mathbb{N}}(\mathbf{2})$ the small multirelations category
- Category $D'(2)$ is apparently the Sugihara algebra S4. Explain it.
- $\mathbf{G}_L(\mathbf{Set})$, connected to Dialectica Petri nets (2025), uses two multirelations. Other concurrency models?
- Linear logic superpower games — the original insight, still open after forty years, etc...

Summary

1. Five objects, one of them odd, and three inequalities that describe the whole construction.
2. Tensor, internal hom and negation computed by hand — and negation is already a genuine involution.
3. ! needs one comonad in $\mathbf{D}$ and two in relay in $\mathbf{G}$. Downstairs the free monoid vanishes, but $\mathbf{D}(H)$ has two genuine modalities — one from contraction, one adjoint to the diagonal embedding.
4. $\mathcal{P}\,\mathbf{D}(\mathbf{Set}) \cong \mathbf{D}'(\mathbf{2})$. Up to poset reflection, the small model **is** the theory.
5. Two dials, one turned so far. Most of the grid is blank.

Pick a cell.

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