Lax amalgamation and equational consequence

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Getting started: some model theory

Let $\mathbf{A}$ be any first-order structure over a language $\mathcal{L}$.

We define a new first-order structure $\mathbf{A}'$ over a new language $\mathcal{L}'$, obtained by adding a fresh constant symbol c_a for every $a \in A$.

Let $\mathbf{A}'$ be the obvious expansion of $\mathbf{A}$ to a structure over the new language $\mathcal{L}'$. The diagram of $\mathbf{A}$ is the set of all atomic sentences and negations of atomic sentences in $\mathcal{L}'$ that hold in $\mathbf{A}'$.

The positive diagram is obtained similarly by just taking the atomic sentences (not the negations of them).

Getting started: some model theory

Classical fact (see e.g. Chang-Keisler)

1. Let $\mathbf{A}$ and $\mathbf{B}$ be structures and let $f : A \to B$ be a function. The map f is an embedding of $\mathbf{A}$ into $\mathbf{B}$ if and only if the induced structure $(\mathbf{B}, \{f(a)\}_{a \in A})$ is a model of the diagram of $\mathbf{A}$.
2. There is a homomorphism from $\mathbf{A}$ into $\mathbf{B}$ if and only if some expansion of $\mathbf{B}$ is a model of the positive diagram of $\mathbf{A}$.

The amalgamation property

Now suppose that $\mathbf{A}$ is a substructure of the structures $\mathbf{B}$ and $\mathbf{C}$. An amalgam of the span $\langle \mathbf{A} \hookrightarrow \mathbf{B}, \mathbf{A} \hookrightarrow \mathbf{C} \rangle$ is a model of the union of the diagrams of $\mathbf{B}$ and $\mathbf{C}$.

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$$\begin{tikzcd}
 & \mathbf{B} \\
\mathbf{A} \arrow[ur, hook, "\phi_1"] \arrow[dr, hook, "\phi_2"'] & \\
 & \mathbf{C}
\end{tikzcd}$$

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$$\begin{tikzcd}
 & \mathbf{B} \arrow[dr, hook, dashed, "\psi_1"] & \\
\mathbf{A} \arrow[ur, hook, "\phi_1"] \arrow[dr, hook, "\phi_2"'] & & \mathbf{D} \\
 & \mathbf{C} \arrow[ur, hook, dashed, "\psi_2"'] &
\end{tikzcd}$$

The positive amalgamation property

Consider the span $\langle \phi_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}, \phi_2 \colon \mathbf{A} \hookrightarrow \mathbf{C} \rangle$. We say that a pair of homomorphisms $\langle \psi_1 \colon \mathbf{B} \to \mathbf{D}, \psi_2 \colon \mathbf{C} \to \mathbf{D} \rangle$ is a positive amalgam of the span if $\psi_1\phi_1 = \psi_2\phi_2$ is injective.

$$
\begin{tikzcd}
 & \mathbf{B} \arrow[dr, dashed, "\psi_1"] & \\
\mathbf{A} \arrow[ur, hook, "\phi_1"] \arrow[dr, hook, "\phi_2"'] & & \mathbf{D} \\
 & \mathbf{C} \arrow[ur, dashed, "\psi_2"'] &
\end{tikzcd}
$$

This talk: positive amalgamation in algebraic structures.

Amalgamation and interpolation

From now on, we will restrict the structures we deal with to algebras.

Amalgamation in the TACL community is often associated with interpolation:

$$\alpha \vdash \beta \Rightarrow \exists \delta(\mathsf{var}(\delta) \subseteq \mathsf{var}(\alpha) \cap \mathsf{var}(\beta), \alpha \vdash \delta, \delta \vdash \beta)$$

Descends from Maksimova's work on Heyting algebras, where amalgamation for varieties of HAs is equivalent to deductive/Craig interpolation for the associated superintutionistic logic.

In more generality, amalgamation is better associated with the Robinson property and the logic-to-algebra bridge is better given in terms of equational consequence.

Recall that if K is a class of similar algebras, we define the equational consequence $\models_{\mathsf{K}}$ by

$$\Sigma \models_{\mathsf{K}} \varepsilon \iff \bar{\varepsilon} \in \mathrm{Cg}_{\mathbf{F}(X)}(\bar{\Sigma}),$$

where X is some set of variables containing all those appearing in $\Sigma \cup \{\varepsilon\}$, $\mathbf{F}(X)$ is the free algebra on X, and $\bar{\cdot}$ denotes the projection of the absolutely free algebra on X to $\mathbf{F}(X)$.

Amalgamation and interpolation

Recall that a variety V has the Robinson property if for any $\Sigma \subseteq \mathrm{Eq}(\bar{x}, \bar{y})$ and $\Pi \cup \{\varepsilon\} \subseteq \mathrm{Eq}(\bar{y}, \bar{z})$ satisfying

1. $\Sigma \models_{\mathsf{V}} \delta \iff \Pi \models_{\mathsf{V}} \delta$ for all $\delta \in \mathrm{Eq}(\bar{y})$,
2. $\Sigma \cup \Pi \models_{\mathsf{V}} \varepsilon$,

then we have $\Pi \models_{\mathsf{V}} \varepsilon$.

Theorem (Metcalfe-Montagna-Tsinakis 2014, older folklore?)

A variety V has the Robinson property if and only if it has the amalgamation property.

The RP implies deductive interpolation (and sometimes conversely), and interpolation/RP for logics is often best seen as coming from the equational version for associated classes of algebras.

Equational logic for positive amalgamation

This talk generalizes the classic set-up to positive amalgamation, getting analogues of the Robinson property, deductive interpolation, and so on.

Like many topics in interpolation, this was already studied for IPC/Heyting algebras and some modal logics by Maksimova.

But there are a lot of surprises in the general situation.

Presents some interesting questions about amalgamation generally.

Part I: The Positive Robinson Property

Recall that a variety V has the Robinson property if for any $\Sigma \subseteq \mathrm{Eq}(\bar{x}, \bar{y})$ and $\Pi \cup \{\varepsilon\} \subseteq \mathrm{Eq}(\bar{y}, \bar{z})$ satisfying

1. $\Sigma \models_{\mathsf{V}} \delta \iff \Pi \models_{\mathsf{V}} \delta$ for all $\delta \in \mathrm{Eq}(\bar{y})$,
2. $\Sigma \cup \Pi \models_{\mathsf{V}} \varepsilon$,

then we have $\Pi \models_{\mathsf{V}} \varepsilon$.

The Positive Robinson Property

A variety V has the positive Robinson property (or PRP) if whenever $\Sigma \subseteq \mathrm{Eq}(\bar{x}, \bar{y})$, $\Pi \subseteq \mathrm{Eq}(\bar{y}, \bar{z})$, and and $\varepsilon \in \mathrm{Eq}(\bar{y})$ satisfy

1. $\Sigma \models_{\mathsf{V}} \delta \iff \Pi \models_{\mathsf{V}} \delta$ for all $\delta \in \mathrm{Eq}(\bar{y})$,
2. $\Sigma \cup \Pi \models_{\mathsf{V}} \varepsilon$,

then $\Pi \models_{\mathsf{V}} \varepsilon$.

Positive Robinson Property

Theorem (Doyle-F.-Krawczyk 2026):

A variety V has the positive Robinson property if and only if V has the positive amalgamation property.

Proof idea: Proof of the usual (non-positive) version factors through the Pigozzi property, and similarly there is a positive version.

Positive Pigozzi property: A variety V has the positive Pigozzi property when for all $\Theta \in \mathrm{Con}(F_{\mathsf{V}}(\bar{x}, \bar{y}))$ and $\Psi \in \mathrm{Con}(F_{\mathsf{V}}(\bar{y}, \bar{z}))$, if Θ and Ψ have the same restriction to $F_{\mathsf{V}}(\bar{y})$, then there is some congruence Φ on $F_{\mathsf{V}}(\bar{x}, \bar{y}, \bar{z})$ so that Φ restricted to $F_{\mathsf{V}}(\bar{y})$ is the same as the restrictions of Θ, Ψ and both $\Psi \subseteq \Phi \cap F_{\mathsf{V}}(\bar{x}, \bar{y})$ and $\Theta \subseteq \Phi \cap F_{\mathsf{V}}(\bar{y}, \bar{z})$ hold (regular Pigozzi property has equality).

The rest goes as expected.

Maehara interpolation

There are also 'positive' versions of some classical strong interpolation properties.

Strong deductive interpolation property/Maehara interpolation property:

$$\begin{aligned}\Gamma, \Sigma \vdash \beta \Rightarrow \exists \Delta \text{ such that } & \\ & \mathsf{var}(\Delta) \subseteq \mathsf{var}(\Gamma) \cap \mathsf{var}(\Sigma, \beta), \\ & \Gamma \vdash \Delta, \text{ and} \\ & \Sigma, \Delta \vdash \beta\end{aligned}$$

Classical result: A variety has MIP if and only if it has the transferable injections property = CEP + AP.

Restricted interpolation property

The restricted interpolation property just modifies the MIP by demanding that β is in the variables common to Γ, Σ:

$$
\begin{aligned}
\Gamma, \Sigma \vdash \beta \Rightarrow \exists \Delta \text{ such that}\\
&\mathrm{var}(\Delta) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma, \beta),\\
&\Gamma \vdash \Delta, \text{ and}\\
&\Sigma, \Delta \vdash \beta
\end{aligned}
$$

Transferable injections and its positive variant

$$
\begin{tikzcd}
 & \mathbf{B} \arrow[dr, dashed, hook, "\psi_1"] & \\
\mathbf{A} \arrow[ur, hook, "\phi_1"] \arrow[dr, "\phi_2"'] & & \mathbf{D} \\
 & \mathbf{C} \arrow[ur, dashed, hook, "\psi_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{B} \arrow[dr, dashed, hook, "\psi_1"] & \\
\mathbf{A} \arrow[ur, hook, "\phi_1"] \arrow[dr, "\phi_2"'] & & \mathbf{D} \\
 & \mathbf{C} \arrow[ur, dashed, "\psi_2"'] &
\end{tikzcd}
$$

A class K has the positive transferable injections property if for every (singly injective) space $\langle \phi_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}, \phi_2 \colon \mathbf{A} \to \mathbf{C} \rangle$ there is an pair of homomorphisms $\langle \psi_1 \colon \mathbf{B} \hookrightarrow \mathbf{D}, \psi_2 \colon \mathbf{C} \to \mathbf{D} \rangle$ in K so that the restriction of ψ_2 to $\phi_2[A]$ is an embedding.

Restricted interpolation property algebraically

Theorem (Doyle-F.-Krawczyk 2026):

Let V be any variety. TFAE.

1. V has the restricted interpolation property.
2. V has both congruence extension property and the positive amalgamation property.
3. V has the positive transferable injections property.

Part II: Positive Amalgamation

Essential embeddings

Let **A**, **B**, **C** be algebras. An embedding $\varphi\colon \mathbf{A} \to \mathbf{B}$ is essential if for each $\psi\colon \mathbf{B} \to \mathbf{C}$, whenever $\psi \circ \phi$ is an embedding we have also that ψ is an embedding.

Equivalently, this holds if every non-identity congruence of **B** restricts to a non-identity congruence of $\varphi[\mathbf{A}]$.

We say that a span $\langle \phi_1, \phi_2 \rangle$ is singly essential if ϕ_2 is an essential embedding and a class K of similar algebras has the singly essential AP if each singly essential span in K has an amalgam in K.

Essential extensions and the AP

Theorem (F.-Santschi 2024):

Let V be a variety.

1. Let $\langle \phi_1 \colon \mathbf{A} \hookrightarrow \mathbf{B}, \phi_2 \colon \mathbf{A} \hookrightarrow \mathbf{C} \rangle$ be a singly essential span in V such that $\mathbf{B} \in \mathsf{V}_{\mathsf{SI}}$. Then $\langle \phi_1, \phi_2 \rangle$ has an amalgam in V if and only if it has an amalgam in V_{SI}.
2. If V has the congruence extension property and $\mathsf{V}_{\mathsf{FSI}}$ is closed under homomorphic images and subalgebras, then V has the amalgamation property if and only if every single essential span in $\mathsf{V}_{\mathsf{FSI}}$ has an amalgam in $\mathsf{V}_{\mathsf{FSI}}$.

Essential extensions and the AP

The previous theorem has been (sometimes should have been) very useful in some of the main results on AP in recent years.

- F.-Santschi APAL 2025: Classification of all varieties of BL-algebras with the AP.
- F.-Metcalfe-Santschi JSL 2026: Uncountably many varieties of idempotent semilinear residuated lattices with the AP, only 60 commutative varieties.

Doubly essential AP

Definition:

We say that a span $\langle \phi_1, \phi_2 \rangle$ is doubly essential if both ϕ_1 and ϕ_2 are essential embeddings. A class K of algebras has the (doubly) essential AP if every doubly essential span in K has an amalgam in K.

Positive amalgamation and subdirectly irreducibles

Theorem (Doyle-F.-Krawczyk 2026):

Let V be any variety.

1. If V has the positive amalgamation property, then V_{SI} has the doubly essential amalgamation property.
2. If V has the congruence extension property and V_{SI} has the doubly essential amalgamation property, then V has the positive amalgamation property.

N.B.: In fact, an amalgam of a doubly essential span in V_{SI} can WLOG be taken to be doubly essential itself.

Proof of 1: Suppose $\langle \mathbf{A} \hookrightarrow \mathbf{B}, \mathbf{A} \hookrightarrow \mathbf{C} \rangle$ is a doubly essential span in V_{SI}.

By PAP, the span can be completed with homomorphisms $\langle \psi_1 \colon \mathbf{B} \to \mathbf{D}, \psi_2 \colon \mathbf{C} \to \mathbf{D} \rangle$ so that $\psi_1\phi_1 = \psi_2\phi_2$ is injective. But by the definition of essentiality, then ψ_1, ψ_2 are embeddings.

Taking an appropriate quotient of $\mathbf{D}$, we can reduce this amalgam to an amalgam in V_{SI}.

Part III: Some Familiar Examples

Pseudocomplemented distributive lattices

A pseudocomplemented distributive lattice is a structure of the form $\langle A, \wedge, \vee, \neg, 0, 1\rangle$, where the $\neg$-free reduct is a bounded lattice and for all $a, b \in A$,

$$a \wedge b = 0 \iff b \leq \neg a.$$

These form a variety with the CEP.

The lattice of subvarieties forms a countable chain of type $\omega + 1$.

Pseudocomplemented distributive lattices

Grätzer and Lakser have shown that there are exactly five varieties of pseudocomplemented distributive lattices with the AP. By an analysis of essential embeddings on SIs, we show:

Theorem (Doyle-F.-Krawczyk 2026):

For any variety of pseudocomplemented distributive lattices, the AP and the PAP coincide and so there are exactly five varieties of these with the PAP.

Sugihara monoids

Sugihara monoids are idempotent, distributive, involutive residuated lattices.

Marchioni-Metcalfe 2012: There are exactly nine varieties of Sugihara monoids with the AP.

Theorem (Doyle-F.-Krawczyk 2026):

There are exactly twelve varieties of Sugihara monoids with the PAP.

MV-algebras

Di Nola-Lettieri 2000: A variety of MV-algebras has the AP if and only if it is generated by a single totally ordered MV-algebra.

Theorem (Doyle-F.-Krawczyk 2026):

Let V be a variety of MV-algebras. Then V has the PAP if and only if V is generated by a single chain, or else $V = \mathbb{V}(\mathbf{L}_n, \mathbf{L}_{m,\omega})$ where m divides n and $m \neq n$.
(Here $\mathbf{L}_n$ is the n-element MV-chain and $\mathbf{L}_{n,\omega} = \Gamma(\mathbb{Z} \times \mathbb{Z}, (n, 0))$).

Some other examples

Often you can extract failures of PAP from known failures of the AP. Some examples of failures:

- Residuated lattices (Jipsen-Santschi 2025)
- MTL-algebras (Giustarini-Ugolini 2025)
- De Morgan monoids (new, but spiritually due to Urquhart in the early 2000s)

Summing up

The positive AP and its associated syntactic properties are quite close to the traditional variants.

Cast much light on the traditional variants, but are vastly easier to get a handle on due to doubly essential AP.

Could push these characterizations very far even to cases that were hard for the AP: Wajsberg hoops, BL-algebras, idempotent semilinear residuated lattices, and so on.

Thank you!