Topology and Metric

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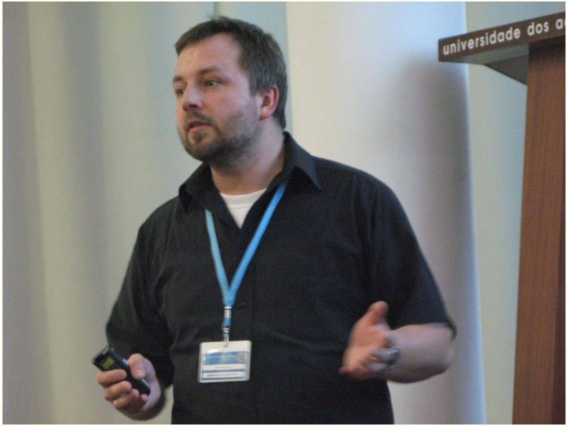
[a]Based on joint work with Marco Abbadini, Maria Manuel Clementino, Pedro Nora, Rui Prezado and Walter Tholen.



Paweł Waszkiewicz (1973 – 2011), Faculty of Theoretical Computer Science, Jagiellonian University, Krákow.

WASZKIEWICZ, PAWEL (2002). **“Quantitative Continuous Domains”.** PhD thesis. School of Computer Science, The University of Birmingham.

"The kinds of structures which actually arise in the practice of geometry and analysis are far from being 'arbitrary' . . . , as concentrated in the thesis that fundamental structures are themselves categories."

LAWVERE, F. WILLIAM (1973). **"Metric spaces, generalized logic, and closed categories".** In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

1. Topology and Order

Theorem (Stone (1938))

$$\mathbf{Spec} \simeq \mathbf{DL}^{\mathrm{op}}.$$

Theorem (Priestley (1970))

$$\mathbf{Priest} \simeq \mathbf{DL}^{\mathrm{op}}.$$

Recall

Priestley space = totally order-disconnected partially ordered compact space.
Spectral space = compact sober T_0-space, the compact opens form a base for the topology.

- Stone, Marshall Harvey (1938). **"Topological representations of distributive lattices and Brouwerian logics".** In: *Časopis pro pěstování matematiky a fysiky* **67**.(1), pp. 1–25.
- Priestley, Hilary A. (1970). **"Representation of distributive lattices by means of ordered Stone spaces".** In: *Bulletin of the London Mathematical Society* **2**.(2), pp. 186–190.
- Priestley, Hilary A. (1972). **"Ordered topological spaces and the representation of distributive lattices".** In: *Proceedings of the London Mathematical Society. Third Series* **24**.(3), pp. 507–530.

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Remark

One can prove directly that **Spec** and **Priest** are isomorphic.

Cornish, William H. (1975). **"On H. Priestley's dual of the category of bounded distributive lattices".** In: *Matematichki Vesnik. New Series* **12**, pp. 329–332.

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One can prove directly that **Spec** and **Priest** are isomorphic.

Remark

$\mathbf{Priest}^{\mathrm{op}}$ is a variety of algebras.

Definition (Nachbin)

An **partial ordered compact space** $(X, \leq, \tau)$ consists of a set X, a partial order $\leq$ on X and a compact topology on X so that the set $\{(x, y) \in X \times X \mid x \leq y\}$ is closed in $X \times X$.

NACHBIN, LEOPOLDO (1950). ***Topologia e Ordem.*** University of Chicago Press.

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Definition

A topological T_0-space is called **stably compact** whenever it is locally compact, sober and stable. A continuous map $f : X \longrightarrow Y$ between stably compact spaces is **spectral** if $f^{-1}(A)$ is compact, for every $A \subseteq Y$ compact lower.

About stable: For a locally compact space X, the following assertions are equivalent.

(i) The way below relation on the lattice of opens is stable under finite intersections.

(ii) The finite intersection of compact lower sets is compact.

(iii) For every ultrafilter $\mathfrak{x}$ on X, the set of all limit points of $\mathfrak{x}$ is irreducible.

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Theorem

$$\mathbf{PosComp} \cong \mathbf{StablyComp}.$$

Gierz, Gerhard, Hofmann, Karl Heinrich, Keimel, Klaus, Lawson, Jimmie D., Mislove, Michael W., and Scott, Dana S. (1980). ***A compendium of continuous lattices.*** Berlin: Springer-Verlag. xx + 371.

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Remark

Priestley space: "cogenerated" by 2 in **PosComp**.
Spectral space: "cogenerated" by 2 in **StablyComp**.

Theorem

$$\begin{tikzcd} \mathbf{PosComp} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & \mathbf{Top} \arrow[l, bend left, "\Sigma"] \end{tikzcd}$$

Here:

- *KX is the space X with topology $\downarrow\tau$.*
- *ΣX is the partially ordered set of prime filters of opens of X with the hull-kernel topology.*

SIMMONS, HAROLD (1982). **"A couple of triples".** In: *Topology and its Applications* **13**.(2), pp. 201–223.

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Moreover, the algebras for $\mathbb{F}_p$ are precisely the stably compact spaces and the algebra homomorphisms are precisely the spectral maps.

Note. A T_0-space is stably compact iff it is locally compact and every ultrafilter (or prime filter of opens) has a smallest convergence point.

$$\begin{tikzcd} F_pX \arrow[rr, bend left, "\alpha"] \arrow[rr, phantom, "\perp"] & & X \arrow[ll] \end{tikzcd}$$

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Remark

Recall also: $\mathbf{ContLat} \cong \mathbf{Top}^{\mathbb{F}}$, where $\mathbb{F}$ is the filter monad of opens on **Top**.

Theorem

SCOTT, DANA (1972). **“Continuous lattices”.** In: *Toposes, Algebraic Geometry and Logic*. Ed. by F. WILLIAM LAWVERE. Springer Berlin Heidelberg, pp. 97–136.

DAY, ALAN (1975). **“Filter monads, continuous lattices and closure systems”.** In: *Canadian Journal of Mathematics* **27**.(1), pp. 50–59.

ESCARDÓ, MARTÍN HÖTZEL (1997). **“Injective spaces via the filter monad”.** In: *Proceedings of the 12th Summer Conference on General Topology and its Applications (North Bay, ON, 1997)*. Vol. 22. Summer, pp. 97–100.

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Remark

One also has **PosComp** $\cong$ **Pos**$^{\mathbb{F}_p}$, where $\mathbb{F}_p$ is the prime filter monad of lower subsets on **Pos**.

Flagg, Robert C. (1997). **“Algebraic theories of compact pospaces”.** In: *Topology and its Applications* **77**.(3), pp. 277–290.

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One also has **PosComp** $\cong$ **Pos**$^{\mathbb{F}_p}$, where $\mathbb{F}_p$ is the prime filter monad of lower subsets on **Pos**.

Recall

CompHaus $\cong$ **Set**$^{\mathbb{U}}$

MANES, ERNEST G. (1969). **"A triple theoretic construction of compact algebras".** In: *Seminar on Triples and Categorical Homology Theory*. Ed. by B. ECKMANN. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 91–118.

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Recall

CompHaus $\simeq$ **Set**$^{\mathbb{U}}$, but also **CompHaus**$^{\mathrm{op}}$ is a variety of (infinitary) algebras.

Theorem

A category is a (possibly infinitary) variety if and only if it is cocomplete, Barr-exact, and it has a regular projective, regular generator.

DUSKIN, JOHN (1969). **"Variations on Beck's tripleability criterion".** In: *Reports of the Midwest Category Seminar III*. Ed. by SAUNDERS MACLANE. Springer Berlin Heidelberg, pp. 74–129.

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PosComp$^{\mathrm{op}}$ *is a quasi-variety of (infinitary) algebras.*

HOFMANN, DIRK, NEVES, RENATO, and NORA, PEDRO (2018). **"Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces".** In: *Theory and Applications of Categories* **33**.(12), pp. 276–295.

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ABBADINI, MARCO (2019). **"The dual of compact ordered spaces is a variety".** In: *Theory and Applications of Categories* **34**.(44), pp. 1401–1439.

Theorem

OrdCH $\cong$ **Ord**$^{\mathbb{U}}$, *here UX is the set of all ultrafilters of X with* $\mathfrak{x} \leq \mathfrak{y}$ *whenever* $\forall A, B\, \exists x, y \,.\, x \leq y$.

Tholen, Walter (2009). **"Ordered topological structures".** In: *Topology and its Applications* **156**.(12), pp. 2148–2157.

Theorem. Let X be a set, $\leq$ an order relation on X and $\alpha \colon UX \longrightarrow X$ a compact Hausdorff topology on X. Then the following assertions are equivalent.

(i) $\{(x, y) \mid x \leq y\} \subseteq X \times X$ is closed.

(ii) $\alpha \colon U(X, \leq) \longrightarrow (X, \leq)$ is monotone.

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OrdCH $\cong$ **Ord**$^{\mathbb{U}}$, *here UX is the set of all ultrafilters of X with $\mathfrak{x} \leq \mathfrak{y}$ whenever $\forall A, B\, \exists x, y \,.\, x \leq y$.*

Theorem

$$\begin{tikzcd} \mathbf{OrdCH} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & \mathbf{Top} \arrow[l, bend left, "M"] \end{tikzcd}$$

- $K(X, \leq, \alpha) = (X, \leq \cdot \alpha \colon UX \nrightarrow X)$, *that is:* $\mathfrak{x} \to x$ *whenever* $\alpha(\mathfrak{x}) \leq x$.

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- $M(X, a \colon UX \nrightarrow X) = (UX, UX \xrightarrow{m_X^\circ} UUX \xrightarrow{Ua} UX, m_X \colon UUX \to UX)$, *that is,*

  $\mathfrak{x} \le \mathfrak{y} \iff \overline{A} \in \mathfrak{y}$, *for all* $A \in \mathfrak{x}$.

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- $K(X, \le, \alpha) = (X, \le \cdot \alpha\colon UX \to\!\!\!\!\!\!\mapsto X)$, *that is:* $\mathfrak{x} \to x$ *whenever* $\alpha(\mathfrak{x}) \le x$.

A relation $a\colon UX \nrightarrow X$ is a topology iff

$$1_X \le a \cdot e_X \quad (\dot{x} \xrightarrow{a} x), \qquad a \cdot Ua \le a \cdot m_X \quad (\mathfrak{X} \xrightarrow{Ua} \mathfrak{x} \xrightarrow{a} x \implies m_X(\mathfrak{X}) \xrightarrow{a} x.)$$

BARR, MICHAEL (1970). **“Relational algebras”.** In: *Reports of the Midwest Category Seminar IV.* Ed. by SAUNDERS MACLANE. Springer Berlin Heidelberg, pp. 39–55.

Definition

The **lower Vietoris functor** $V : \mathbf{Top} \longrightarrow \mathbf{Top}$ sends a topological space X to the space

$$VX = \{A \subseteq X \mid A \text{ is closed}\}$$

of closed subsets of X, with the topology generated by the sets

$$B^{\Diamond} = \{A \in VX \mid A \cap B \neq \varnothing\} \quad (B \subseteq X \text{ open}),$$

and $Vf : VX \longrightarrow VY$ sends A to $\overline{f[A]}$, for $f : X \longrightarrow Y$ in $\mathbf{Top}$.

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Remark

From that we obtain $V\colon \mathbf{StablyComp} \longrightarrow \mathbf{StablyComp}$ and $V\colon \mathbf{CompHaus} \longrightarrow \mathbf{CompHaus}$, and further restrictions to **Priest** and **BooSp**.

Recall:

$$\begin{tikzcd} \mathbf{CompHaus} \arrow[r, bend left, "\text{discrete}"] \arrow[r, phantom, "\perp"] & \mathbf{PosComp} \arrow[l, bend left, "\text{forget}"] \end{tikzcd} \simeq \mathbf{StablyComp}$$

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Vietoris, Leopold (1922). **"Bereiche zweiter Ordnung".** In: *Monatshefte für Mathematik und Physik* **32**.(1), pp. 258–280.

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WYLER, OSWALD (1981). **"Algebraic theories of continuous lattices".** In: *Continuous Lattices*. Ed. by BERNHARD BANASCHEWSKI and RUDOLF-EBERHARD HOFFMANN. Springer-Verlag, pp. 390–413.

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closed set	$= X \longrightarrow 2$
filter (of opens)	$= (UX)^{\mathrm{op}} \longrightarrow 2$

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Compare with: $\mathbf{Set}^{\mathbb{P}} \simeq \mathbf{Ord}^{\mathbb{P}_{\downarrow}} \simeq \mathbf{Ord}^{\mathbb{P}^{\uparrow}}$.

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$\mathbf{Priest}_{\mathbb{V}} \simeq {\mathbf{DL}_{\perp,\vee}}^{\mathrm{op}}$.

CIGNOLI, ROBERTO, LAFALCE, S., and PETROVICH, ALEJANDRO (1991). **“Remarks on Priestley duality for distributive lattices”.** In: *Order* **8**.(3), pp. 299–315.

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and $Vf\colon VX \longrightarrow VY$ sends A to $\overline{f[A]}$, for $f\colon X \longrightarrow Y$ in $\mathbf{Top}$.

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From that we obtain $V\colon \mathbf{StablyComp} \longrightarrow \mathbf{StablyComp}$ and $V\colon \mathbf{CompHaus} \longrightarrow \mathbf{CompHaus}$, and further restrictions to $\mathbf{Priest}$ and $\mathbf{BooSp}$.

Theorem

$$\mathbf{CompHaus}^{\mathbb{V}} \simeq \mathbf{Top}^{\mathbb{F}} \simeq \mathbf{StablyComp}^{\mathbb{V}}.$$

Theorem

$$\mathbf{PriestDist} \simeq {\mathbf{FinSup}_{\mathrm{DL}}}^{\mathrm{op}}.$$

CIGNOLI, ROBERTO, LAFALCE, S., and PETROVICH, ALEJANDRO (1991). **“Remarks on Priestley duality for distributive lattices”.** In: *Order* **8**.(3), pp. 299–315.

2. From ordered sets to metric spaces

Analogy of form

Thinking of an order relation on a set M as a function

$$f\colon M \times M \longrightarrow \{<, >, =\},$$

Hausdorff observes that

> *"Nun steht einer Verallgemeinerung dieser Vorstellung nichts im Wege, und wir können uns denken, daß eine beliebige Funktion der Paare einer Menge definiert, d.h. jedem Paar (a, b) von Elementen einer Menge M ein bestimmtes Element $n = f(a, b)$ einer zweiten Menge N zugeordnet sei. In noch weiterer Verallgemeinerung können wir eine Funktion der Elementtripel, Elementfolgen, Elementkomplexe, Teilmengen u. dgl. von M in Betracht ziehen."*

HAUSDORFF, FELIX (1914). ***Grundzüge der Mengenlehre.*** Leipzig: Veit & Comp. viii + 476.

Analogy of form

Thinking of an order relation on a set M as a function

$$f\colon M \times M \longrightarrow \{<, >, =\},$$

Hausdorff observes that

> *Now there stands nothing in the way of a generalisation of this idea, and we can think of an arbitrary function of pairs of points which associates to each pair (a, b) of elements of a set M a specific element $n = f(a, b)$ of a second set N. Generalising further, we can consider a function of triples, sequences, complexes, subsets, etc.*

HAUSDORFF, FELIX (1914). ***Grundzüge der Mengenlehre.*** Leipzig: Veit & Comp. viii + 476.

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- A metric is a function $d\colon M \times M \longrightarrow \mathbb{R}_0^+ \ldots$
- A topology is a function $\mathcal{N}\colon \mathrm{P}M \times M \longrightarrow \mathbf{2} \ldots$

HAUSDORFF, FELIX (1914). ***Grundzüge der Mengenlehre.*** Leipzig: Veit & Comp. viii + 476.

Analogy of laws

"While listening to a 1967 lecture of Richard Swan, which included a discussion of the relative codimension of pairs of subvarieties, I noticed the analogy between the triangle inequality and a categorical composition law."

$$
\begin{array}{rcc}
\text{In } ([0,\infty],\geqslant): & 0 \geqslant d(x,x), & d(x,y)+d(y,z) \geqslant d(x,z). \\
\text{In } ([0,\infty],\geqslant): & 0 \geqslant d(x,x), & \max(d(x,y),d(y,z)) \geqslant d(x,z). \\
\text{In } (\{\bot,\top\},\implies): & \top \implies [\![x \le x]\!], & ([\![x \le y]\!] \,\&\, [\![y \le z]\!]) \implies [\![x \le z]\!], \\
\text{In } \mathbf{Set} & 1 \longrightarrow \mathbf{C}(x,x), & \mathbf{C}(x,y) \times \mathbf{C}(y,z) \longrightarrow \mathbf{C}(x,z), \\
\text{In } \mathbf{Vec}_k & k \longrightarrow \mathbf{C}(x,x), & \mathbf{C}(x,y) \otimes \mathbf{C}(y,z) \longrightarrow \mathbf{C}(x,z).
\end{array}
$$

LAWVERE, F. WILLIAM (1973). **"Metric spaces, generalized logic, and closed categories".** In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

Theorem

The ultrafilter monad on **Set** *extends to a monad on* **Met** *by putting*

$$U(X,d)=(UX,Ud), \qquad \text{where } Ud(\mathfrak{x},\mathfrak{y})=\sup_{A,B}\inf_{x,y} d(x,y).$$

Here we think of a map $r\colon X\times Y\longrightarrow[0,\infty]$ as an (enriched) relation $r\colon X \nrightarrow Y$.

THOLEN, WALTER (2009). **“Ordered topological structures”.** In: *Topology and its Applications* **156**.(12), pp. 2148–2157.

Theorem

The ultrafilter monad on **Set** *extends to a monad on* **Met** *by putting*

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Definition

We call an algebra for this monad **metric compact Hausdorff space**, and put $\mathbf{MetCH} = \mathbf{Met}^{\mathbb{U}}$.

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We call an algebra for this monad **metric compact Hausdorff space**, and put **MetCH** $=$ **Met**$^{\mathbb{U}}$.

Theorem

For a set X equipped with a metric d and a compact Hausdorff topology $\alpha\colon UX \longrightarrow X$, tfae:

1. $\alpha\colon U(X,d) \longrightarrow (X,d)$ *is metric.*
2. $d\colon X \times X \longrightarrow [0,\infty]$ *is lower semicontinuous.*

HOFMANN, DIRK and REIS, CARLA D. (2018). **"Convergence and quantale-enriched categories".** In: *Categories and General Algebraic Structures with Applications* **9**.(1), pp. 77–138.

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A metric space X is **separated** whenever

$$(0 = X(x,y) \;\&\; 0 = X(y,x)) \implies x = y.$$

Example

Every compact separated metric space is a metric compact Hausdorff space.

Theorem

The ultrafilter monad on **Set** *extends to a monad on* **Met** *by putting*

$$U(X,d) = (UX, Ud), \qquad \textit{where } Ud(\mathfrak{x},\mathfrak{y}) = \sup_{A,B} \inf_{x,y} d(x,y).$$

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2. $d\colon X \times X \longrightarrow [0,\infty]$ *is lower semicontinuous.*

Example

$[0,\infty]$ with distance $\mu(u,v) = v \ominus u$ and the Euclidean topology.

Definition

An **approach space** is a set X together with a function $\delta \colon PX \times X \longrightarrow [0, \infty]$ subject to

1. $\delta(\{x\}, x) = 0$,
2. $\delta(\varnothing, x) = \infty$,
3. $\delta(A \cup B, x) = \min\{\delta(A, x), \delta(B, x)\}$,
4. $\delta(A^{(\varepsilon)}, x) + \varepsilon \geqslant \delta(A, x)$;

LOWEN, ROBERT (1989). **“Approach spaces: a common supercategory of TOP and MET”.** In: *Mathematische Nachrichten* **141**.(1), pp. 183–226.

LOWEN, ROBERT (1997). ***Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad.*** Oxford Mathematical Monographs. Oxford University Press. x + 253.

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Theorem

A map $a\colon UX \times X \longrightarrow [0,\infty]$ is the convergence of an approach space if and only if

$$0 \geqslant a(\dot{x},x) \quad \text{and} \quad Ua(\mathfrak{X},\mathfrak{x}) + a(\mathfrak{x},x) \geqslant a(m_X(\mathfrak{X}),x).$$

CLEMENTINO, MARIA MANUEL and HOFMANN, DIRK (2003). **“Topological Features of Lax Algebras”.** In: *Applied Categorical Structures* **11**.(3), pp. 267–286.

Theorem

$$\begin{tikzcd} \mathbf{MetCH} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & \mathbf{App} \arrow[l, bend left, "M"] \end{tikzcd} \qquad \mathbf{MetCH} \simeq \mathbf{App}^{\mathbb{U}}$$

$$K(X, \leq, \alpha) = (X, \leq \cdot \alpha\colon UX \nrightarrow X), \quad M(X, a\colon UX \nrightarrow X) = (UX, UX \xrightarrow{m_X^\circ} UUX \xrightarrow{Ua} UX, m_X\colon UUX \to UX).$$

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$$\begin{tikzcd} \mathbf{MetCH} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & \mathbf{App} \arrow[l, bend left, "M"] \end{tikzcd} \qquad \mathbf{MetCH} \simeq \mathbf{App}^{\mathbb{U}}$$

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Theorem

A T_0 approach space is stably compact if and only if it is sober, $+$-exponentiable and stable.

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$$\begin{tikzcd} \mathbf{MetCH} \arrow[rr, bend left, "K"] \arrow[rr, phantom, "\top"] & & \mathbf{App} \arrow[ll, bend left, "M"] \end{tikzcd} \qquad \mathbf{MetCH} \simeq \mathbf{App}^{\mathbb{U}}$$

$$K(X, \leq, \alpha) = (X, \leq \cdot \alpha \colon UX \nrightarrow X), \quad M(X, a \colon UX \nrightarrow X) = (UX, UX \xrightarrow{m_X^\circ} UUX \xrightarrow{Ua} UX, m_X \colon UUX \to UX).$$

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Theorem

*We have a functor $V \colon$ **App** $\longrightarrow$ **App** sending X to the approach space $\{X \to [0, \infty]\}$, V is part of a monad $\mathbb{V}$ and restricts to **MetCH**, moreover, $\mathbf{MetCH}^{\mathbb{V}} \simeq \mathbf{App}^{\mathbb{F}}$.*

HOFMANN, DIRK (2014). **“The enriched Vietoris monad on representable spaces”.** In: *Journal of Pure and Applied Algebra* **218**.(12), pp. 2274–2318.

3. Duality theory

Motivation

Before: $\mathbf{PriestDist}^{\mathrm{op}} \simeq \mathbf{FinSup}_{\mathbf{DL}}$, here $X \longmapsto \mathbf{Priest}(X, 2^{\mathrm{op}})$

Now: $C\colon \mathbf{PosCompDist}^{\mathrm{op}} \longrightarrow \mathbf{FinSup}_{\mathbf{DL}}$, here $X \longmapsto \mathbf{PosComp}(X, 2^{\mathrm{op}})$

HOFMANN, DIRK and NORA, PEDRO (2018). **"Enriched Stone-type dualities".** In: *Advances in Mathematics* **330**, pp. 307–360.

HOFMANN, DIRK and NORA, PEDRO (2023). **"Duality theory for enriched Priestley spaces".** In: *Journal of Pure and Applied Algebra* **227**.(3), pp. 1–32.

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Three kinds of metric spaces

- $([0,\infty], \geqslant, +, 0) \cong ([0,1], \leq, *, 1)$.
- $([0,\infty], \geqslant, \max, 0) \cong ([0,1], \leq, \wedge, 1)$.
- $([0,1], \geqslant, \oplus, 0) \cong ([0,1], \leq, \odot, 1)$, here $u \odot v = \max\{0, u+v-1\}$ is the Łukasiewicz sum.

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Definition

$[0,1]$-**FinSup** is the category of finitely cocomplete separated metric spaces and morphisms.

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Theorem

The category $[0,1]$-**FinSup** *has a bimorphism representing monoidal structure.*

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Now: $C \colon \mathbf{PosCompDist}^{\mathrm{op}} \longrightarrow \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})$, here $X \longmapsto \mathbf{PosComp}(X, [0,1]^{\mathrm{op}})$

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Theorem

For $*$ *and* $\odot$ *on* $[0,1]$.

1. *The functor* $C\colon \mathbf{PosCompDist}^{\mathrm{op}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})$ *is fully faithful.*

Theorem

For $*$ and $\odot$ on $[0,1]$.

1. *The functor $C\colon \mathbf{PosCompDist}^{\mathrm{op}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})$ is fully faithful.*
2. *$\varphi\colon X \longrightarrow\!\!\!\!\!\!\circ\;\; Y$ in $\mathbf{PosCompDist}$ is a function of and only if $C\varphi$ preserves the monoid structure.*

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3. *The functor* $C\colon \mathbf{PosCompDist}^{\mathrm{op}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\mathbf{FinSup})$ *restricts to a fully faithful functor* $C\colon \mathbf{PosComp}^{\mathrm{op}} \longrightarrow \mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})$.

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Remark

$C\colon \mathbf{PosCompDist}^{\mathrm{op}} \longrightarrow \mathsf{LaxMon}([0,1]_{\wedge}\text{-}\mathbf{FinSup})$ is not full.

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Remark

$C\colon \mathbf{PosCompDist}^{\mathrm{op}} \longrightarrow \mathsf{LaxMon}([0,1]_{\wedge}\text{-}\mathbf{FinSup})$ is not full.

Remark

$[0,1]\text{-}\mathbf{FinSup}$ and $\mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})$ are quasi-varieties.

Definition

A metric (in the three senses) compact Hausdorff space X is called **Priestley** whenever the cone $(f : X \longrightarrow [0, 1]^{\mathrm{op}})_f$ is point-separating and initial.

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For $*$ and $\odot$, $C : [0, 1]\text{-}\mathbf{PriestDist}^{\mathrm{op}} \longrightarrow [0, 1]\text{-}\mathbf{FinSup}$ is fully faithful.

Theorem

For $\odot$, this functor restricts to a fully faithful functor $C : [0, 1]\text{-}\mathbf{Priest}^{\mathrm{op}} \longrightarrow [0, 1]\text{-}\mathbf{FinLat}$. Moreover, C is right adjoint and preserves $\aleph_1$-filtered colimits.

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Question

Is $[0,1]\text{-}\mathbf{Priest}^{\mathrm{op}}$ a variety? And what about $\mathbf{MetCH}^{\mathrm{op}}$?

Definition

A category **C** is **regular** whenever **C** is finitely complete, has pushouts of kernel pairs and regular epimorphisms are pullback-stable.

Setting

We consider here $([0, \infty], \geqslant, +, 0)$.

ABBADINI, MARCO and HOFMANN, DIRK (2025). **"Barr-coexactness for metric compact Hausdorff spaces".** In: *Theory and Applications of Categories* **44**.(6), pp. 196–226.

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Setting

We consider here $([0, \infty], \geqslant, +, 0)$.

Theorem

$\mathbf{MetCH}_{\mathrm{sep}}^{\mathrm{op}}$ *is a regular category.*

Proof.

The epimorphisms in $\mathbf{MetCH}_{\mathrm{sep}}$ are precisely the surjective morphisms, and the regular monomorphisms are precisely the embeddings.
Moreover, the pushout of a regular monomorphism along any morphims is a regular monomorphism. $\square$

Definition

$\mathbf{C}$ is **Barr-exact** whenever it is regular and every internal equivalence relation is effective.

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Definition

A binary corelation on X is called respectively reflexive, symmetric, transitive provided that:

$$
\begin{tikzcd}
 & X+X \arrow[dl, two heads, "\binom{q_0}{q_1}"'] \arrow[dr, two heads, "\binom{1_X}{1_X}"] & \\
S \arrow[rr, dashed, "d"] & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X+X \arrow[dl, two heads, "\binom{q_0}{q_1}"'] \arrow[dr, two heads, "\binom{q_1}{q_0}"] & \\
S \arrow[rr, dashed, "s"] & & S
\end{tikzcd}
$$

reflexivity

symmetry

$$
\begin{tikzcd}
X \arrow[r, "q_1"] \arrow[d, "q_0"'] & S \arrow[d, "\lambda_1"] \\
S \arrow[r, "\lambda_0"'] & P \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
\implies
\begin{tikzcd}
 & X+X \arrow[dl, two heads, "\binom{q_0}{q_1}"'] \arrow[dr, "\binom{\lambda_0\circ q_0}{\lambda_1\circ q_1}"] & \\
S \arrow[rr, dashed, "t"] & & P
\end{tikzcd}
$$

transitivity

Theorem

Every equivalence corelation in $\mathbf{MetCH}_{\mathrm{sep}}$ *is effective; in particular,* $\mathbf{MetCH}_{\mathrm{sep}}^{\mathrm{op}}$ *is Barr-exact.*

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Theorem ("Urysohn")

Let X be a metric compact Hausdorff space and $A, B \subseteq X$ closed subsets. Put $r := X(A, B)$. Then there exists a continuous 1-Lipschitz function $f\colon X \longrightarrow [0, r]$ with $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(r)$.

MATOUŠKOVÁ, EVA (2000). **"Extensions of Continuous and Lipschitz Functions".** In: *Canadian Mathematical Bulletin* **43**.(2), pp. 208–217.

BEN YAACOV, ITAÏ (2013). **"Lipschitz functions on topometric spaces".** In: *Journal of Logic and Analysis*, pp. 1–21.

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Theorem

For all $a \leq b$, $[a, b]$ *(with the asymmetric metric) is a regular injective object in* $\mathbf{MetCH}_{\mathrm{sep}}$. *Moreover, the product*

$$\prod_{n \in \mathbb{N}} [0, n]$$

(where $[0, n]$ *carries the asymmetric metric) is a regular cogenerator in* $\mathbf{MetCH}_{\mathrm{sep}}$.

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Every equivalence corelation in $\mathbf{MetCH}_{\mathrm{sep}}$ is effective; in particular, $\mathbf{MetCH}_{\mathrm{sep}}^{\mathrm{op}}$ is Barr-exact.

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$$\prod_{n \in \mathbb{N}} [0, n]$$

(where $[0, n]$ carries the asymmetric metric) is a regular cogenerator in $\mathbf{MetCH}_{\mathrm{sep}}$.

Theorem

The dual of $\mathbf{MetCH} = [0, 1]$-$\mathbf{Priest}$ a variety of $\aleph_1$-ary algebras.

4. Compact metric spaces as exponentiable metric compact spaces

(joint work with Rui Prezado)

Definition

Let $\mathbf{C}$ be a category with finite products. An object X of $\mathbf{C}$ is called **exponentiable** whenever $- \times X$ has a right adjoint $[X, -]$.

The category $\mathbf{C}$ is **Cartesian closed** if every object is exponentiable.

Recall that $- \times X \dashv [X, -]$ means that, for all Y, Z:

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Assume that $\mathbf{C}$ comes with a faithful functor $|-| : \mathbf{C} \longrightarrow \mathbf{Set}$ represented by the terminal object. Then $|[X, Y]| \cong \mathbf{C}(X, Y)$

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\begin{tikzcd}
{|[X, Y] \times X|} \arrow[d, "\cong"'] \arrow[dr, "counit"] & \\
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\end{tikzcd}
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\begin{tikzcd}
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DAY, BRIAN J. and KELLY, G. MAX (1970). **“On topologically quotient maps preserved by pullbacks or products”.** In: *Proceedings of the Cambridge Philosophical Society* **67**.(03), pp. 553–558.

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Example

A topological space is exponentiable in **Top** iff it is core-compact.

Theorem

A compact Hausdorff space is exponentiable in **CompHaus** *iff it is discrete (finite).*

Cagliari, Francesca and Mantovani, Sandra (1991). **“Local homeomorphisms as the exponentiable morphisms in compact Hausdorff spaces”.** In: *Topology and its Applications* **41**.(3), pp. 263–272.

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If X is discrete, then $[X, Y] = \prod_{x \in X} Y$.

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If X is discrete, then $[X, Y] = \prod_{x \in X} Y$.
Assume now that X is exponentiable in **CompHaus**, and consider $x \in X$. For every $U \in \mathcal{O}(x)$, put

$$U^{\#} = \{f : X \to [0,1] \mid f(x) = 0, U^{\complement} \subseteq f^{-1}(1)\} \subseteq [X, [0,1]].$$

By Urysohn's lemma, $U^{\#} \neq \varnothing$; and $\{U^{\#} \mid U \in \mathcal{O}(x)\}$ is a (proper) filter base on $[X, [0,1]]$. Take an ultrafilter $\mathfrak{x}$ containing it, then $\mathfrak{x} \to f \in [X, [0,1]]$. Then

$$\begin{aligned} \mathrm{ev}_x(\mathfrak{x}) &\to 0 && \text{hence } f(x) = 0, \\ \mathrm{ev}_y(\mathfrak{x}) &\to 1 && \text{hence } f(y) = 1 \text{ for } x \neq y. \end{aligned}$$

Hence $\{x\}$ is open in X. □

Theorem

Let X be a metric space. Then the following assertions are equivalent.

(i) *X is exponentiable in* **Met**.

(ii) *X is totally quasi-convex.*

The product metric is given by the max-**metric**:

$$X \times Y((x,y),(x',y')) = \max\{X(x,x'), Y(y,y')\}$$

for all $x, x' \in X$ and $y, y' \in Y$.

Definition

A metric space X is **totally quasi-convex** whenever, for all $x_1, x_2 \in X$ and all $u_1, u_2 \in [0,\infty]$ such that $d(x_1,x_2) = u_1 + u_2$, for all $\varepsilon > 0$, there exists $z \in X$ with $d(x_1,z) \leq u_1 + \varepsilon$ and $d(z,x_2) \leq u_2 + \varepsilon$.

CLEMENTINO, MARIA MANUEL and HOFMANN, DIRK (2006). **“Exponentiation in V-categories”.** In: *Topology and its Applications* **153**.(16), pp. 3113–3128.

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Corollary

A complete metric space is exponentiable in **Met** *iff it is totally convex (take "$\varepsilon = 0$" above).*

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Unfortunately, this is wrong, as shown by Carlyle Stewart (University of Stellenbosch).

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A metric compact Hausdorff space is totally quasi-convex iff it is totally convex.

Remark

For a metric compact Hausdorff space X:

- The closed balls are closed but the open balls need not be open.

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Let X be in **MetCH**. *Then (1) $\implies$ (2) $\implies$ (3).*

(1) *X is exponentiable on* **MetCH**.

(2) *For all Y, $[X, Y]$ is a closed subspace of the product space $\prod_{x \in X} Y$.*

(3) *The open balls are open in the topology of X*

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For a separated metric compact Hausdorff space:

$$X \text{ is exponentiable in } \mathbf{MetCH} \implies X \text{ is compact metric.}$$

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Theorem

For a separated metric compact Hausdorff space:

$$X \text{ is exponentiable in } \textbf{MetCH} \iff X \text{ is compact metric and totally convex.}$$

Remark

For metric spaces X and Y, we consider the Cartesian product $X \times Y$ with the **plus-metric**

$$d((x,y),(x',y')) := X(x,x) + Y(y,y').$$

This operation makes **Met** and **MetCH** symmetric monoidal categories, and **Met** is even closed with $[X,Y]_+ = \mathbf{Met}(X,Y)$ and

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THE PLUS-METRIC

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Theorem

For a separated metric compact Hausdorff space X:

$$X \text{ is } \oplus\text{-exponentiable in } \mathbf{MetCH} \iff X \text{ is compact metric.}$$

The setting

Besides $+$, we consider a commutative quantale structure $\otimes$ on $[0, \infty]$ with neutral element 0 such that, for all $u_1, u_2, v_1, v_2 \in [0, \infty]$,

$$(u_1 + u_2) \otimes (v_1 + v_2) \leq (u_1 \otimes v_1) + (u_2 \otimes v_2).$$

Then **Met** (respectively **MetCH**) is symmetric monoidal (if $\otimes \colon [0, \infty] \times [0, \infty] \longrightarrow [0, \infty]$ is lower semicontinuous), with

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Examples

- We may consider $\otimes = +$ and $\otimes = \max$.
- For $1 < p < \infty$, we may consider $u \otimes_p v := \sqrt[p]{u^p + v^p}$, thanks to the Minkowski inequality:

$$\sqrt[p]{(u_1 + v_1)^p + (u_2 + v_2)^p} \leq \sqrt[p]{u_1^p + u_2^p} + \sqrt[p]{v_1^p + v_2^p}.$$

Theorem

A metric space X is $\otimes$-exponentiable in **Met** *if and only if, for all $x_1, x_2 \in X$ and all $u_1, u_2 \in [0, \infty]$,*

$$\inf_{z \in X} \big((d(x_1, z) \otimes u_1) + (d(z, x_2) \otimes u_2)\big) \leq d(x_1, x_2) \otimes (u_1 + u_2).$$

In this case, the right adjoint $[X, -]_\otimes$ of $- \otimes X$ is given by $[X, Y]_\otimes = \mathbf{Met}(X, Y)$ with

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Proposition

For a compact separated metric space X:

$$X \text{ is } \otimes\text{-exponentiable in } \mathbf{Met} \implies X \text{ is } \otimes\text{-exponentiable in } \mathbf{MetCH}.$$

Proposition

Assume that $\otimes\colon [0,\infty] \times [0,\infty] \longrightarrow [0,\infty]$ *is continuous. Then a metric compact Hausdorff space which is* $\otimes$*-exponentiable in* **MetCH** *is also* $\otimes$*-exponentiable in* **Met**.

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Recall the condition: $\inf_{z\in X}\left((d(x_1,z)\otimes u_1)+(d(z,x_2)\otimes u_2)\right)\leq d(x_1,x_2)\otimes(u_1+u_2)$.

Proof.

Let $x_1,x_2\in X$ and all $u_1,u_2\in[0,\infty]$ and consider

$$\begin{aligned}
\varphi_1\colon X &\longrightarrow [0,\infty], & x &\longmapsto d(x_1,-),\\
\varphi_2\colon X &\longrightarrow [0,\infty], & x &\longmapsto d(x_1,-)\otimes u_1,\\
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$$(u_1+u_2)\otimes d(x_1,x_2)\ge\inf_{z\in X}\left((d(x_1,z)\otimes u_1)+(d(z,x_2)\otimes u_2)\right).$$

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For a separated metric compact Hausdorff space X:

$$\otimes\text{-exponentiable in } \mathbf{MetCH} \iff X \text{ is compact metric and } \otimes\text{-exponentiable in } \mathbf{Met}.$$

Corollary

In particular, a separated metric compact Hausdorff space is

- *$\oplus$-exponentiable iff it is compact metric.*
- *exponentiable iff it is compact metric and totally convex.*

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