A logic for weighted Markov chains

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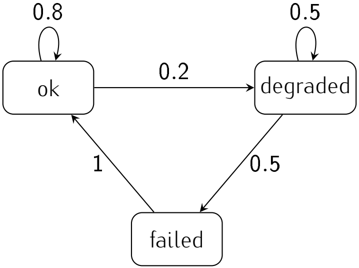

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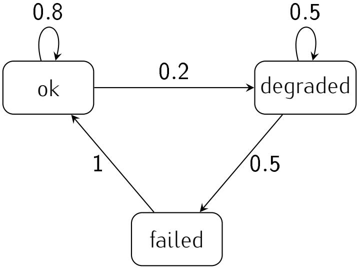

Formally: a set of states X and, for each state x, a probability distribution $\alpha(x)$ on X giving the probability of moving to another state.

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Chains recording this data are what we'll be calling weighted Markov chains.

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The engine

A Stone-type duality between weighted Markov chains and modal ℓ-groups, that extends the Furber-Mardare-Mio duality.

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We want a simple many-valued modal logic whose modality is probabilistic rather than normal, with a good algebraic semantics – and complete with respect to Markov chains.

The roadmap

$$\begin{tikzcd}
\text{Markov–Łukasiewicz logic} \arrow[rr, leftrightarrow, "\text{alg. completeness}"] & & \text{MMV modal MV-algebras} \arrow[d, leftrightarrow, "\text{Mundici } \Gamma"] \\
\text{NMarkov}_a \subseteq \text{Markov}_a \text{ weighted Markov chains} \arrow[rr, leftrightarrow, "\textit{this work}"] & & \text{(complete) unital Archimedean modal } \ell\text{-groups} \\
\text{Markov} \cong \text{Markov}_0 \text{ (FMM) Markov chains} \arrow[u, hook, "\zeta \equiv 0"'] \arrow[rr, leftrightarrow, "\text{FMM duality}"] & & \text{complete unital Archimedean modal Riesz spaces}
\end{tikzcd}$$

Background

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The algebraic interpretation

The logic

Reminders: ℓ-groups, Riesz spaces, ideals

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- Mundici's Γ: unital ℓ-groups $\simeq$ MV-algebras. The functor, on objects, is $\Gamma(G, u) = [0, 1_G]$.

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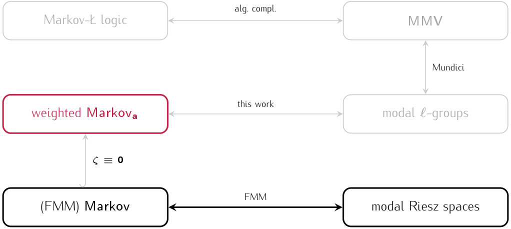

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An a-space (X, ζ) is (arithmetically) normal if it is compact and, for any two distinct points $x, y \in X$, there exist two disjoint open neighborhoods U and V of x and y respectively, such that, for every $z \in X \setminus (U \cup V)$ we have $\zeta(z) = 0$.

The main example

Let $\mathsf{den}\colon \mathbb{R} \to \mathbb{N}$ be defined as $\mathsf{den}(z) = \begin{cases} 0 & z \in \mathbb{R} \setminus \mathbb{Q} \\ \text{denominator of } z & \text{otherwise} \end{cases}$.

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A function $f\colon (X, \zeta_X) \to (Y, \zeta_Y)$ is an a-map if it is continuous and it decreases denominators, i.e. $\zeta_Y(f(x))$ divides $\zeta_X(x)$ for all $x \in X$.

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If X is compact, $C_a(X)$ is an ℓ-group that has 1_X as strong unit, is Archimedean and is (norm-)complete.

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$C_a \dashv \mathrm{Max}_a$

The adjunction restricts to a duality between the category of complete unital ℓ-groups and the one of normal arithmetic spaces.

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We do the same over an a-space. Let (X, ζ) be a compact Hausdorff a-space, and set

$$R_a(X) \coloneqq \{F_{|C_a(X)} \mid F \in R(X)\}.$$

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- For any a-map $f\colon (X, \zeta_X) \to (Y, \zeta_Y)$ in the category $\mathsf{CHaus_a}$ of compact Hausdorff a-spaces, the map

$$R_a(f) : (R_a(X), \mathsf{den}_R) \to (R_a(Y), \mathsf{den}_R)$$

such that $R_a(f)(F)(g) := F(g \circ f)$ for all $F \in R_a(X)$ and $g \in C_a(Y)$ is an a-map.

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Two useful results:

- For any a-space $(X, \zeta) \in \mathsf{CHaus_a}$ the space $R_a(X)$ is a normal a-space.
- For any $(X, \zeta) \in \mathsf{Norm_a}$ and for any $F \in R_a(X)$ there exists a unique $F' \in R(X)$ that extends F.

A discrete example

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Consider the discrete space $X = \{x_1, x_2, x_3, x_4\}$, and the function $\zeta\colon X \to \mathbb{N}$ defined as $\zeta(x_1) = 0$, $\zeta(x_2) = 4$, $\zeta(x_3) = \zeta(x_4) = 2$.

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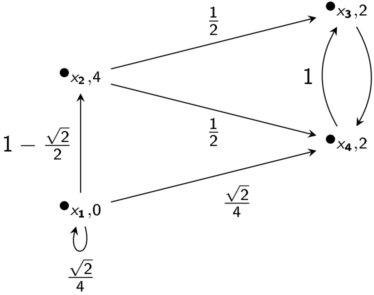

Indeed, for any $f \in C_a(X)$ we have that $\mathsf{den}(\alpha(x_i)(f)) | \zeta(x_i)$, since $\mathsf{den}(f(x_i))$ is a divisor of $\zeta(x_i)$.

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So our framework properly extends FMM's — their category is the $\zeta \equiv 0$ case of ours.

R. Furber, R. Mardare, M. Mio, *Probabilistic Logics Based on Riesz Spaces*, LMCS 16(1:6) (2020).

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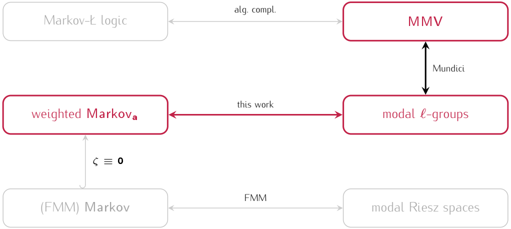

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Any weighted Markov chain induces a modal ℓ-group!

Modal ℓ-groups

A modal ℓ-group is an algebra $(G, 1_G, \Diamond)$, where $(G, 1_G)$ is an ℓ-group with distinguished positive element 1_G and $\Diamond : G \to G$ is a positive group homomorphism such that $\Diamond(1_G) \leq 1_G$.

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We define $C_\Diamond : \mathsf{Markov_a} \to \mathsf{MAu\ell Grp}$ as follows.

- for any object $((X, \zeta), \alpha)$ in $\mathsf{Markov_a}$, $C_\Diamond((X, \zeta), \alpha) := (C_a(X), 1_X, \Diamond_\alpha)$, where $\Diamond_\alpha(f)(x) = \alpha(x)(f)$ for all $x \in X$,
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- on arrows is the usual pre-composition.

Morally, $\Diamond_\alpha(f)(x)$ is the expected value of f after one step from x.

From groups to chains (the idea)

Given an Archimedean modal unital ℓ-group $(G, 1_G, \Diamond)$ and $x \in \mathrm{Max}_a(G) \subseteq \mathbb{R}^G$, the map $x \circ \Diamond$ is a positive homomorphism with $(x \circ \Diamond)(1_G) \leq 1$.

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This defines a functor $\mathrm{Max}_\Diamond : \mathbf{MAu\ell Grp} \to \mathbf{Markov_a}$ (pre-composition on arrows).

more detail

The duality

The functors $C_\Diamond$ and $\mathrm{Max}_\Diamond$ form a contravariant adjunction $C_\Diamond \dashv \mathrm{Max}_\Diamond$, which restricts to a duality:

$$\begin{tikzcd}
\mathsf{Markov_a} \arrow[rr, bend left, "C_\Diamond"] & \perp & \mathsf{MAu\ell Grp} \arrow[ll, "\mathrm{Max}_\Diamond"] \\
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Proof idea: the unit and counit of the Abbadini–Marra–Spada adjunction are arrows in our categories.

Modal MV-algebras: the bridge to a logic

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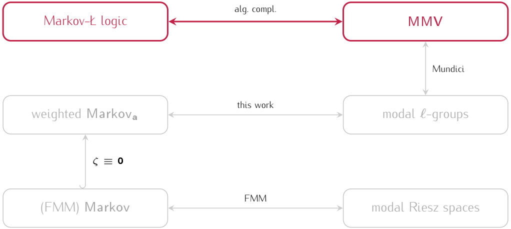
So our weighted chains carry a propositional, many-valued, modal syntax.

Background

Weighted Markov chains

The algebraic interpretation

The logic



Markov-Łukasiewicz logic

In the language $\mathcal{L}$ given by **Var**, the connectives of Łukasiewicz logic, and a unary connective $\Diamond$, the set **MFm** of well-formed formulas is defined recursively as follows:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \to \varphi \mid \Diamond\varphi$$

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  - (ML3) $\Diamond(\varphi \oplus \psi) \leftrightarrow \Diamond(\varphi) \oplus \Diamond(\neg\varphi \wedge \psi)$,
- Rules:
  - (MP) from φ and $\varphi \to \psi$ derive ψ,
  - (M) from $(\varphi \to \psi)$ derive $(\Diamond(\varphi) \to \Diamond(\psi))$.

Algebraic completeness

Let $\Gamma \cup \{\varphi\} \subseteq \mathbf{MFm}$. Then,

Theorem 1 (algebraic completeness)

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Moreover, adding (M) as an axiom does not work!

Indeed

$$\not\models_{\text{MMV}} (\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi).$$

An example

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Then $v(\Diamond(p)) = \Diamond(v(p)) = \Diamond\left(\frac{1}{2}, \frac{2}{5}\right) = \left(\frac{12}{25}, \frac{2}{5}\right)$ and

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On the other hand:

$$v(\Diamond(\top) \to \Diamond(p)) = (1,1) \to \left(\frac{12}{25}, \frac{2}{5}\right) = \left(\frac{12}{25}, \frac{2}{5}\right).$$

Therefore,

$$v((\top \to p) \to (\Diamond(\top) \to \Diamond(p))) = \left(\tfrac{49}{50}, 1\right) \neq (1,1).$$

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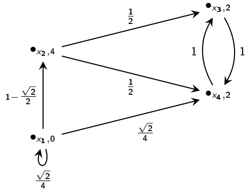
Valuation in a Markov model: an example

Take the weighted Markov chain $((X,\zeta),\alpha)$ from before, and $p \in \mathsf{Var}$ with $f_p(x_1) = 0,\ f_p(x_2) = 1,\ f_p(x_3) = 0,\ f_p(x_4) = 1.$



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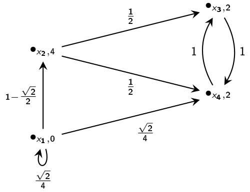

$t^{\mathfrak{M}}_{\Diamond p}(x) = \alpha(x)(f_p)$ is the expected value of p after one step:

$$t_{\Diamond p}(x_1) = \frac{\sqrt{2}}{4}\cdot 0 + \left(1-\frac{\sqrt{2}}{2}\right)\cdot 1 + \frac{\sqrt{2}}{4}\cdot 1 = 1-\frac{\sqrt{2}}{4},$$

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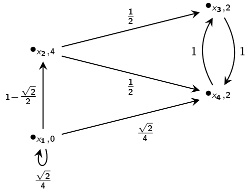

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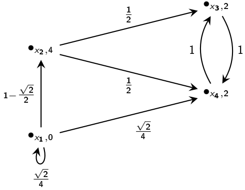
$$t_{\Diamond p}(x_3) = 1\cdot 1 = 1,$$

$$t_{\Diamond p}(x_4) = 1\cdot 0 = 0.$$

It is an a-map: irrational at x_1 ($\mathsf{den} = 0 = \zeta(x_1)$), and $\mathsf{den}(\frac{1}{2}) = 2 \mid 4 = \zeta(x_2)$.

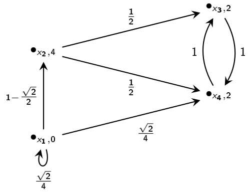
Valuation in a Markov model: an example (cont.)

Recall $f_p = (0,1,0,1)$ and $t_{\Diamond p} = \left(1 - \frac{\sqrt{2}}{4},\ \frac{1}{2},\ 1,\ 0\right)$ on (x_1, x_2, x_3, x_4).



Valuation in a Markov model: an example (cont.)

Recall $f_p = (0,1,0,1)$ and $t_{\Diamond p} = \left(1 - \frac{\sqrt{2}}{4},\ \frac{1}{2},\ 1,\ 0\right)$ on (x_1, x_2, x_3, x_4).



Evaluating $\Diamond p \to p$ pointwise, with $a \to b = \min(1, 1 - a + b)$:

$$t_{\Diamond p \to p} = t_{\Diamond p} \to f_p = \left(\frac{\sqrt{2}}{4},\ 1,\ 0,\ 1\right).$$

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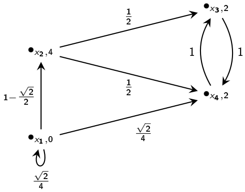

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So $\mathfrak{M}, x_2 \Vdash \Diamond p \to p$, but $\mathfrak{M} \nVdash \Diamond p \to p$ (it fails e.g. at x_3).

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So, consider the extension **aML** of the Markov-Łukasiewicz logic obtained by adding the following rule:

(A) from $n\varphi \to \psi$ for all $n \in \mathbb{N}$ derive $\neg\varphi$.

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The main point of this proof is that the Lindenbaum-Tarski algebra of **aML** is Archimedean.

This allows us to use the adjunction between aMMV and $\mathsf{Markov_a}$ to prove the right-to-left direction.

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In this case, we show that there exists an adjunction between $\mathsf{NMarkov_a}$ and Markov and get the result from the Corollary above.

Summing up

One modality, a finite language, one infinitary rule — and algebraic completeness without it

- **Theorem 1.** the base calculus is complete w.r.t. modal MV-algebras — *finitary*.
- **Theorem 2.** adding rule (A) gives completeness w.r.t. (weighted) Markov chains over a *finite language*.

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Put it differently: is the theory of modal Riesz spaces a conservative extension of the theory of modal ℓ-groups?

Thank you for your attention!

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Markov–Łukasiewicz}\\ \text{logic}\end{array}} \arrow[r, leftrightarrow, "\text{alg. completeness}"] & \boxed{\begin{array}{c}\mathsf{MMV}\\ \text{modal MV-algebras}\end{array}} \arrow[d, leftrightarrow, "\text{Mundici } \Gamma"] \\
\boxed{\begin{array}{c}\mathsf{NMarkov_a} \subseteq \mathsf{Markov_a}\\ \text{weighted Markov chains}\end{array}} \arrow[r, leftrightarrow, "\textit{this work}"] & \boxed{\begin{array}{c}\text{(complete) unital Archimedean}\\ \text{modal } \ell\text{-groups}\end{array}} \\
\boxed{\begin{array}{c}\mathsf{Markov} \cong \mathsf{Markov_0}\\ \text{(FMM) Markov chains}\end{array}} \arrow[u, hook, "\zeta \equiv 0"] \arrow[r, leftrightarrow, "\text{FMM duality}"] & \boxed{\begin{array}{c}\text{complete unital Archimedean}\\ \text{modal Riesz spaces}\end{array}}
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Backup: from groups to chains (full construction)

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It is not difficult (but not trivial) to see that $\alpha_\Diamond$ is continuous and decreases denominators; hence $(\mathrm{Max}_a(G), \alpha_\Diamond)$ is a weighted Markov chain. back