Prolegomena zu einer jeden künftigen Modallogik, die als Wissenschaft wird auftreten können

Intuitionistic strict implication, sharply and flatly

Tadeusz Litak
Based on several joint papers (next slide)

Past: UJ, JAIST, UvA, Birkbeck UoL, Leicester, FAU, UniNa
Present: Freelance
Future: Università degli Studi di Padova

TACL 2026

The basis of this talk

- L. and Visser, 2018. "Lewis meets Brouwer: Constructive strict implication". Indagationes Mathematicae 29. A special issue "L.E.J. Brouwer, fifty years later", pp. 36–90. `https://arxiv.org/abs/1708.02143`
- de Groot, L. and Pattinson, 2026. "Gödel-McKinsey-Tarski and (not quite) Blok-Esakia for Heyting-Lewis Implication". A corrected full version of LiCS 2021 paper. `https://arxiv.org/abs/2105.01873`
- de Groot and L., 2026. "Relational semantics for flat Heyting-Lewis logic". In: Proceedings of AiML 2026, EPTCS 447, 2026, pp. 445-463. `https://arxiv.org/abs/2603.28402`
- L. and Visser, 2021. "Lewisian fixed points I: Two incomparable constructions". In: Volume honouring Dick de Jongh's 80th birthday. Outstanding Contributions to Logic. Springer. `http://arxiv.org/abs/1905.09450`
- Visser and L., 2024. Lewis and Brouwer meet Strong Löb. `https://doi.org/10.48550/arXiv.2404.11969`

Roots, bloody roots

What's in a name?

- We could call our subject today the implication/calculus of
  - Heyting-Lewis
  - Heyting-Lewis-Visser
  - Heyting-Lewis-Visser-Iemhoff
  - Heyting-Lewis-Visser-Iemhoff-Celani-Jansana
  - *any of the above options, but with Brouwer replacing Heyting and possibly permuting names*
  - Heyting-weak-Heyting (HwH)
- or just simply constructive/intuitionistic strict implication
- iP (Iemhoff and coauthors): because preservativity in HA
- iA (L. + Visser), also because arrows in FP

But... why Lewis?

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{\prec} \quad \varphi, \psi ::= \top \mid \bot \mid p \mid \varphi \to \psi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \prec \psi$$

- $\prec$ is the strict implication of Clarence Irving Lewis

  who is not C.S. Lewis, David Lewis or Lewis Carroll

  His earliest series of papers on the subject: 1912–1915

  1918: *A Survey of Symbolic Logic*, University of California Press

  1932 (with C. H. Langford): *Symbolic Logic*, Dover

- $\Box\varphi$ is then definable as $\top \prec \varphi$
- Over the Classical Propositional Calculus (CPC) and in the presence of an additional axiom (di), the converse holds too:
  $\varphi \prec \psi$ is $\Box(\varphi \to \psi)$, i.e., $\top \prec (\varphi \to \psi)$

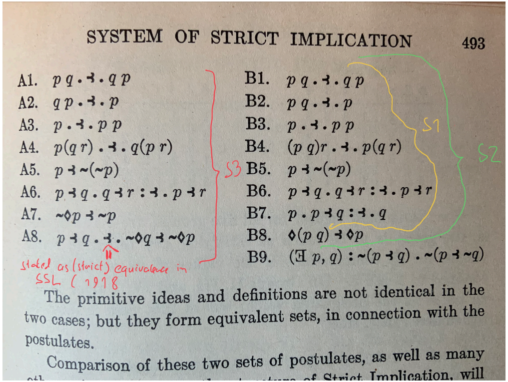
- Lewis' early system with classical negation developed in *A Survey of Symbolic Logic* (1918) just collapsed $\strictif$ to $\to$
- Lewis' later summary:

  > *In developing the system, I had worked for a month to avoid this principle, which later turned out to be false. Then, finding no reason to think it false, I sacrificed economy and put it in.*

- In Appendix II to *Symbolic Logic* (1932), Lewis was more cautious, creating several *lines of retreat* (Parry): S1–S3
- Interestingly, C.I. Lewis was curious about non-classical logics

  mostly the Łukasiewicz logic: possible implication connectives in finite valued logics

  - A detailed discussion in *Symbolic Logic*, 1932
  - A paper on "Alternative Systems of Logic", *The Monist*, same year
- We found just one reference where he mentions (favourably!) Brouwer's criticism of excluded middle
- But found no indication he was aware of Kolmogorov, Heyting, Glivenko ...



SYSTEM OF STRICT IMPLICATION 493

A1. $p\,q . \strictif . q\,p$
A2. $q\,p . \strictif . p$
A3. $p . \strictif . p\,p$
A4. $p(q\,r) . \strictif . q(p\,r)$
A5. $p \strictif \sim(\sim p)$
A6. $p \strictif q . q \strictif r : \strictif . p \strictif r$
A7. $\sim\Diamond p \strictif \sim p$
A8. $p \strictif q . \strictif . \sim\Diamond q \strictif \sim\Diamond p$

stated as (strict) equivalence in SSL (1918

S3

B1. $p\,q . \strictif . q\,p$
B2. $p\,q . \strictif . p$
B3. $p . \strictif . p\,p$
B4. $(p\,q)r . \strictif . p(q\,r)$
B5. $p \strictif \sim(\sim p)$
B6. $p \strictif q . q \strictif r : \strictif . p \strictif r$
B7. $p . p \strictif q : \strictif . q$
B8. $\Diamond(p\,q) \strictif \Diamond p$
B9. $(\exists\, p, q) : \sim(p \strictif q) . \sim(p \strictif \sim q)$

S1

S2

The primitive ideas and definitions are not identical in the two cases; but they form equivalent sets, in connection with the postulates.

Comparison of these two sets of postulates, as well as many oth... of Strict Implication, will

- Lewis rules are

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \qquad \varphi \strictif \psi}{\psi} \qquad \frac{\varphi \strictiff \psi}{\chi(\varphi) \strictiff \chi(\psi)}$$

  plus uniform substitution, of course
- Among B1–B7 only B7 underivable in $\mathsf{HLC}^\flat$ (defined below)
- Curry-Howardly, it's the axiom of higher-order arrows/classical arrows with apply, i.e., (strong) monads!

Axiomatization for $\strictif$: $\mathsf{HLC}^\flat$ vs. $\mathsf{HLC}^\sharp$

Axioms and rules of IPC:

$$\varphi \to (\psi \to \varphi)$$
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

$$\bot \to \varphi$$

$$(\varphi \wedge \psi) \to \varphi$$
$$(\varphi \wedge \psi) \to \psi$$
$$\varphi \to (\psi \to (\varphi \wedge \psi))$$

$$\varphi \to (\varphi \vee \psi)$$
$$\psi \to (\varphi \vee \psi)$$
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$$

Axioms and rules of the minimal system $\mathsf{HLC}^\flat$:

(a.k.a. iA^- or iP^- or HL^-)

Those of IPC plus:

$$(\mathsf{tra}) \quad (\varphi \dashv\!\!\!\!3\ \psi) \wedge (\psi \dashv\!\!\!\!3\ \chi) \rightarrow (\varphi \dashv\!\!\!\!3\ \chi)$$

transitivity of $\dashv\!\!\!\!3$

$$(\mathsf{k_a}) \quad (\varphi \dashv\!\!\!\!3\ \psi) \wedge (\varphi \dashv\!\!\!\!3\ \chi) \rightarrow (\varphi \dashv\!\!\!\!3\ (\psi \wedge \chi))$$

normality=normality in the second coördinate

$$(\mathsf{n_a})\ \frac{\varphi \rightarrow \psi}{\varphi \dashv\!\!\!\!3\ \psi.}$$

binary generalization of necessitation

not only implies congruentiality, but also anti-monotonicity in the first coördinate

Axioms and rules of the full system $\mathsf{HLC}^\sharp$:

(a.k.a. iA or iP or HL)

All the axioms and rules of IPC and $\mathsf{HLC}^\flat$ and

$$(\mathsf{di}) \quad ((\varphi \dashv\!\!\!\!3\ \chi) \wedge (\psi \dashv\!\!\!\!3\ \chi)) \rightarrow ((\varphi \vee \psi) \dashv\!\!\!\!3\ \chi).$$

should implication be anti-multiplicative in the first coördinate?

Running this axiom system via the AAL machinery yields
flat Heyting-Lewis algebras:

(Lewisian Heyting Algebra Expansions, i.e, L-HAEs?)

Heyting algebras plus:

$$\mathsf{CTra} \quad (\varphi \prec\!\!\!\!\mathsf{3}\ \psi) \wedge (\psi \prec\!\!\!\!\mathsf{3}\ \chi) \leq \varphi \prec\!\!\!\!\mathsf{3}\ \chi$$

$$\mathsf{CK_a} \quad (\varphi \prec\!\!\!\!\mathsf{3}\ \psi) \wedge (\varphi \prec\!\!\!\!\mathsf{3}\ \chi) = \varphi \prec\!\!\!\!\mathsf{3}\ (\psi \wedge \chi)$$

$$\mathsf{CId} \qquad \varphi \prec\!\!\!\!\mathsf{3}\ \varphi = \top$$

The class of sharp Heyting-Lewis algebras:

(Lewisian Heyting Algebra with Operators, i.e., L-HAO? Lewis-Brouwer algebras? Heyting-weak-Heyting algebras?)

all the equalities above plus

$$\mathsf{CDi} \quad (\varphi \prec\!\!\!\!\mathsf{3}\ \chi) \wedge (\psi \prec\!\!\!\!\mathsf{3}\ \chi) = (\varphi \vee \psi) \prec\!\!\!\!\mathsf{3}\ \chi.$$

The $\prec\!\!\!\!\mathsf{3}$-free reduct: Heyting algebras
The $\rightarrow$-free reduct:

weak Heyting algebras of Celani and Jansana

Thinking outside the $\Box$

- Recall $\Box\varphi$ is defined as $\top \strictif \varphi$
- We have $\vdash^{\flat} \Box(\varphi \to \psi) \to (\varphi \strictif \psi)$
- But also $\vdash^{\flat} (\varphi \strictif \psi) \to (\Box\varphi \to \Box\psi)$
- This yields the two extreme $\Box$-interpretations of $\strictif$
- The first one inspires the sharp Kripke semantics
- The second one inspires the flat Kripke semantics
- Note: it does not mean that either of these semantics turns either implication into equivalence!

Special cases

- Normal modal logics over CPC
  = varieties of modal algebras (MAs, BAOs)
  (Boolean algebras with a single unary operator)
  Too many applications and references to discuss
- Normal modal logics over IPC (with $\Box$ only!)
  = varieties of HAMs or HA(d)Os
  (Heyting algebras with a modality =
  Heyting algebras with a single unary (dual) operator)
  Again, too many applications and references to discuss
- superintinuitionistic (intermediate) propositional logics
  = varieties of Heyting algebras

But there is much else ...

Intuitionistic ⥽

- Metatheory of arithmetic
  (Σ^0_1–)preservativity for a theory T extending HA
  (or any foundational theory of your choice)
  Generalizes classical interpretability/(Π^0_1–)conservativity, contrapositively
- Functional programming: (Haskell) arrows of John Hughes
  (versus applicative functors/idioms of McBride/Patterson)
  A series of papers (Lindley & Wadler & Yallop, Atkey, Jacobs & Heunen & Hasuo . . .) and numerous Haskell libraries
- Proof theory of guarded (co)recursion
  Nakano and more recently Clouston&Goré
- Analysis of intuitionistic Kripke frames
  sharply: generalizing defining conditions of profunctors/weakening relations
  flatly: "global" ⥽-clause, with no constraints on the modal relation
- Intuitionistic Epistemic Logic of Entailments (IELE)
  Generalizing Artemov& Protopopescu's IEL: recently with Jim de Groot

Sharp systems? Handle with care

- Arithmetically: (di) holds in the preservativity logic of HA, but not in its PA-counterpart

  A sufficient condition to hold: closure under q-realisability

  Recall $\varphi \rhd \psi$ is $\neg\psi \prec \neg\varphi$, classically
- Computationally: (di) models arrows with choice, while arrows obtained using automata or functions on infinite streams do not have choice
- Kripke-style: (di) prevents the combination of $\prec$ with a meaningful $\Diamond$ à la Fischer-Servi/Simpson
- Philosophically: sharp $\prec$ models necessarily true entailments, while flat $\prec$ models entailments of necessities

Sharp frames and an amazing Curry-Howard coincidence

Sharp semantics

- A sharp frame is a tuple $(W, \leq, R)$ s.t.

$$w \leq vRu \quad \Longrightarrow \quad wRu$$

modal accessibility closed under precomposing with intuitionistic order

Necessary to ensure persistence: see the next slide

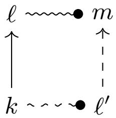
- A sharp model is a tuple $(W, \leq, R, V)$ consisting of a sharp frame and an upward-closed valuation of atoms

$$w \Vdash \varphi \strictif \psi \iff \text{for all } v \in W \text{ (if } wRv \text{ and } v \Vdash \varphi \text{ then } v \Vdash \psi\text{)}$$

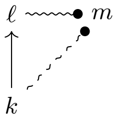
- It is a "world-by-world" or "local" interpretation
- Classical-looking: no need to mention $\leq$ in $\strictif$-clause
- When $\leq$ is the identity relation (discrete), this yields

$$\Box(\varphi \to \psi) \leftrightarrow \varphi \strictif \psi$$

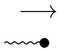
Four frame conditions (occuring since the 1980's)



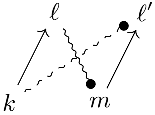
□-p
(persistence for □)



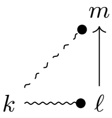
prefixing
(persistence for $-\!\!3$)



intuitionistic $\leq$
modal R



mix/brilliancy
profunctors/weakening rels.



upward-flat
(postfixing)

$\Longleftarrow$ both equivalent
in presence of □-p ,
collapsing $-\!\!3$ to □

- brilliancy obtains naturally in, e.g., Stone-Jónsson-Tarski for □
- . . . but $-\!\!3$ can feel it! $\Longrightarrow$ collapse of $-\!\!3$ to □
- Over prefixing (or $-\!\!3$-frames) $\Box(\varphi \to \psi)$ implies $\varphi \mathrel{-\!\!3} \psi$, but not the other way around

Idioms are oblivious, arrows are meticulous, monads are promiscuous

Sam Lindley, Philip Wadler and Jeremy Yallop

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Abstract

We revisit the connection between three notions of computation: Moggi's *monads*, Hughes's *arrows* and McBride and Paterson's *idioms* (also called *applicative functors*). We show that idioms are equivalent to arrows that satisfy the type isomorphism $A \rightsquigarrow B \simeq 1 \rightsquigarrow (A \to B)$ and that monads are equivalent to arrows that satisfy the type isomorphism $A \rightsquigarrow B \simeq A \to (1 \rightsquigarrow B)$. Further, idioms embed into arrows and arrows embed into monads.

Keywords: applicative functors, idioms, arrows, monads

$$\begin{tikzcd}
\text{idioms} \arrow[d, bend left] & & \text{monads} \arrow[d, bend left] \\
\begin{array}{c}\text{static arrows}\\ (A \rightsquigarrow B \simeq 1 \rightsquigarrow (A \to B))\end{array} \arrow[u, bend left] \arrow[r, hook] & \text{arrows} \arrow[r, hook] & \begin{array}{c}\text{higher-order arrows}\\ (A \rightsquigarrow B \simeq A \to (1 \rightsquigarrow B))\end{array} \arrow[u, bend left]
\end{tikzcd}$$

Fig. 1. Idioms, arrows and monads

$\rightsquigarrow$ here is our $\multimap$

ENTCS 2011, proceedings of MSFP 2008

Many programs and libraries involve components that are "function-like", in that they take inputs and produce outputs, but are not simple functions from inputs to outputs … [S]uch "notions of computation" defin[e] a common interface, called "arrows". … Monads … serve a similar purpose, but arrows are more general. In particular, they include notions of computation with static components, independent of the input, as well as computations that consume multiple inputs.

Ross Paterson

- I'd suggest calling FP arrows strong arrows
- They satisfy in addition the axiom

$$(\mathsf{s_a}) \quad (\varphi \to \psi) \to (\varphi \mathrel{-\!\!\!3} \psi)$$

- …or, equivalently, $(\mathsf{s}_\Box) \quad \varphi \to \Box\varphi$
- Why "equivalently"?

  After all, many $\Box$-principles not equivalent to $\mathrel{-\!\!\!3}$-counterparts

$$\begin{aligned} \varphi \to \psi &\leq \Box(\varphi \to \psi) \\ &\leq \varphi \mathrel{-\!\!\!3} \psi \end{aligned}$$

- $(\mathsf{s}_\Box)$ forces R to be contained in $\leq$:
  - rather degenerate in the classical case

    only three consistent logics of (disjoint unions of) singleton(s)
  - and yet intuitionistically you have a whole CS zoo

    type inhabitation of idioms, arrows, strong monads/PLL …

    plus superintuitionistic logics as a degenerate case

    also a recent proposal for Intuitionistic Epistemic Logic (**IEL**) by Artemov and Protopopescu, which we generalize to **IELE**

Haskell arrows as proposed by John Hughes

```
class Arrow a where
  arr   :: (b -> c) -> a b c
  (>>>) :: a b c ->  a c d -> a b d
  first :: a b c ->  a (b, d) (c, d)
```

$$(\mathsf{s_a}) \qquad (\beta \to \gamma) \to (\beta \multimap \gamma)$$

$$(\mathsf{tra}) \quad (\beta \multimap \gamma) \wedge (\gamma \multimap \delta) \to (\beta \multimap \delta)$$

$$(\mathsf{k'_a}) \qquad (\beta \multimap \gamma) \to ((\beta \wedge \delta) \multimap (\gamma \wedge \delta))$$

Where's (di) ?

```
class Arrow arr => ArrowChoice arr where
  (|||) :: arr a c -> arr b c ->
           arr (Either a b) c
```

$$\textsf{(di)} \quad ((\alpha \multimap \gamma) \wedge (\beta \multimap \gamma)) \to ((\alpha \vee \beta) \multimap \gamma)$$

Arrows with choice?

+ function spaces
+ applicative functors/ arrows with delay
+ monads (see the next slide)
+ Kleisli arrows
+ co-Kleisli arrows if monad distributes over coproducts
+ list processors
  the interleaving pattern in the output is modelled on the interleaving of the input

- automata transforming elements of type a to elements of type b that satisfy the isomorphism $\mathsf{A}\, a\, b \cong a \to b \times (\mathsf{A}\, a\, b)$
- functions on infinite streams

Monads

(recall they allow decomposing $\beta \multimap \gamma$ as $\beta \to \Box\gamma$)

```
class Arrow  => ArrowApply a where
 app :: a (a b c, b) c
```

$$(\mathsf{app_a}) \qquad ((\beta \multimap \gamma) \wedge \beta) \multimap \gamma$$

Recall it's precisely Lewis's B7!
The only S2 axiom underivable in $\mathsf{HLC}^\sharp$...

A1.	$p\,q \,.\!\prec\!.\, q\,p$	B1.	$p\,q \,.\!\prec\!.\, q\,p$
A2.	$q\,p \,.\!\prec\!.\, p$	B2.	$p\,q \,.\!\prec\!.\, p$
A3.	$p \,.\!\prec\!.\, p\,p$	B3.	$p \,.\!\prec\!.\, p\,p$
A4.	$p(q\,r) \,.\!\prec\!.\, q(p\,r)$	B4.	$(p\,q)r \,.\!\prec\!.\, p(q\,r)$
A5.	$p \prec \sim(\sim p)$	B5.	$p \prec \sim(\sim p)$
A6.	$p \prec q \,.\, q \prec r :\prec.\, p \prec r$	B6.	$p \prec q \,.\, q \prec r :\prec.\, p \prec r$
A7.	$\sim\Diamond p \prec \sim p$	B7.	$p \,.\, p \prec q :\prec.\, q$
A8.	$p \prec q \,.\!\prec\!.\, \sim\Diamond q \prec \sim\Diamond p$	B8.	$\Diamond(p\,q) \prec \Diamond p$
		B9.	$(\exists\, p, q) : \sim(p \prec q) \,.\, \sim(p \prec \sim q)$

S3

S1

S2

stated as (strict) equivalence in SSL (1918

The primitive ideas and definitions are not identical in the two cases; but they form equivalent sets, in connection with the postulates.

Comparison of these two sets of postulates, as well as many

Sharp results for sharp systems

- Extensive investigation of semantics for extensions of $\mathsf{HLC}^\sharp$

  A very Dutch affair, especially at the turn of the century: Visser, Iemhoff, de Jongh, Zhou …

  an early overview in L. & Visser 2018
- Originally in the context of preservativity logic

  hence the name $\mathsf{iP}^-/\mathsf{iP}$
- Esakia-style duality: based on Celani & Jansana
- Gödel-McKinsey-Tarski translation and transfer results

  de Groot, L., Pattinson, LiCS 2021
- An analogue of Fine's subframe fmp over $(\mathsf{p_a})$: $(p \prec q) \to \Box(p \prec q)$
- GMT works quite well, allowing transfer of completeness, fmp, decidability …
- Blok-Esakia less so

  A mistake in our LiCS 2021, found by Cheng Liao and Nick Bezhanishvili, corrected on arXiv only
- In the presence of Löb-like axioms, uniform interpolation with strength …

  With Albert Visser, `https://arxiv.org/abs/2404.11969`

Gödel-McKinsey-Tarski and (not-quite) Blok-Esakia

- Generalize the Wolter-Zakharyaschev (1997-98) strategy for unimodal intuitionistic logics with $\Box$
- We extend the Gödel-(McKinsey-Tarski) translation to $\mathcal{L}_{\multimap}$:

$$t_{\multimap}(\varphi \multimap \psi) := \Box_{\mathsf{i}}\Box_{\mathsf{m}}(t_{\multimap}\varphi \to t_{\multimap}\psi)$$

- The translation reflects decidability, completeness, fmp. Above

$$(\mathsf{hl}) \qquad \Box_{\mathsf{m}}\varphi \to \Box_{\mathsf{i}}\Box_{\mathsf{m}}\varphi$$

  it also reflects canonicity.

  enough to find one **S4HL**-counterpart with the desired property!

  Note that in **S4HL** we can also simplify $t_{\multimap}(\varphi \multimap \psi)$ to $\Box_{\mathsf{m}}(t_{\multimap}\varphi \to t_{\multimap}\psi)$

- We can use classical modal metatheory

  e.g., the Sahlqvist/SQEMA algorithm for canonicity and completeness

- ... via a suitable notion of "descriptive frames"

  can be casted as an Esakia-style duality, or as a Priestley-style duality

The GMT result in the bimodal syntax

Theorem

Suppose Θ is a canonical extension of $\mathsf{S4} \otimes \mathsf{K4}$ containing (hl) that is closed under forming (R_{m}-cofinal) subframes. Then:

1. *Θ has the finite model property.*
2. *If moreover Θ contains the classical strength axiom*

   $$(\mathsf{s_c}) \qquad \Box_{\mathsf{i}} p \to \Box_{\mathsf{m}} p.$$

   then for any (R_{m}-cofinal) subframe logic $\Gamma \subseteq \mathcal{L}_{\mathsf{m}}$, the logic $\Theta \oplus \Gamma$ has the finite model property.

Corollary

Let Λ be a $\multimap$-logic extending the axiom of transitivity of R

$$(\mathsf{p}) \qquad (\varphi \multimap \psi) \to \Box(\varphi \multimap \psi)$$

1. *If its* S4HL*-counterparts include a canonical logic preserved by forming (cofinal) subframes, Λ itself has the fmp.*
2. *Furthermore, if Λ extends $(\mathsf{s}_\Box)$ and its* S4HL*-counterparts include a logic obtained by extending a canonical (cofinal) subframe logic with a collection of $\mathcal{L}_{\mathsf{m}}$-axioms preserved by R_m-subframes, Λ has the fmp.*

In either case, Λ is decidable whenever finitely axiomatizable.

This covers the (p) axiom itself, the strength axiom, a strong variant of the Löb axiom, the axiom of monads $(\mathsf{app_a})$ ...

However, some creativity is needed ...

Come flatten my world(s)

Flat semantics

A flat model is a tuple $(W, \preceq, R, V)$ such that

- $(W, \preceq)$ is a preordered set
- R is a relation on W
- $V : \text{Prop} \to \mathsf{up}(W, \preceq)$ is a valuation

Flat semantics

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Arrow semantics:

$$\mathfrak{M}, w \Vdash \varphi \multimapinv \psi \quad \text{iff} \quad \text{for all } v \in W \text{ (if } w \preceq v \text{ and } R[v] \subseteq V(\varphi) \text{ then } R[v] \subseteq V(\psi))$$

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Exactly what I promised:

"global" or "successor-state-by-successor-state" $\multimap$-clause,
but with no constraints on R!

Flat semantics

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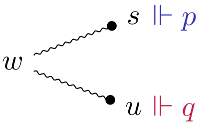
Example:



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Example:

$$\begin{tikzcd}
 & s \Vdash p \\
w \arrow[ur, squiggly] \arrow[dr, squiggly] & \\
 & u \Vdash q
\end{tikzcd}$$

$w \Vdash p \strictif r$ (because $R[w] \not\subseteq V(p)$)

$w \Vdash q \strictif r$ (because $R[w] \not\subseteq V(q)$)

Flat semantics

A flat model is a tuple $(W, \preceq, R, V)$ such that

- $(W, \preceq)$ is a preordered set
- R is a relation on W
- $V : \mathrm{Prop} \to \mathsf{up}(W, \preceq)$ is a valuation

Arrow semantics:

$$\mathfrak{M}, w \Vdash \varphi \strictif \psi \quad \text{iff} \quad \text{for all } v \in W \text{ (if } w \preceq v \text{ and } R[v] \subseteq V(\varphi) \text{ then } R[v] \subseteq V(\psi))$$

Example:

$$\begin{tikzcd}[row sep=small]
 & s \Vdash p \\
w \arrow[ur, squiggly, dash] \arrow[dr, squiggly, dash] & \\
 & u \Vdash q
\end{tikzcd}$$

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$w \nVdash (p \vee q) \strictif r$

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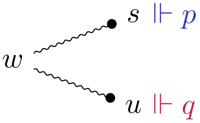
Example:



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$w \Vdash q \strictif r$ (because $R[w] \not\subseteq V(q)$)

$w \nVdash (p \vee q) \strictif r$

So (di) $((\varphi \strictif \chi) \wedge (\psi \strictif \chi)) \to ((\varphi \vee \psi) \strictif \chi)$ is not valid in flat semantics

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Arrow semantics:

$$\mathfrak{M}, w \Vdash \varphi \multimap \psi \quad \text{iff} \quad \text{for all } v \in W \text{ (if } w \preceq v \text{ and } R[v] \subseteq V(\varphi) \text{ then } R[v] \subseteq V(\psi))$$

Example:

$$\begin{tikzcd} v \arrow[r, squiggly] \arrow[d, bend left] & s \Vdash p \\ w \arrow[u, bend left] \arrow[r, squiggly] & u \Vdash q \end{tikzcd}$$

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Arrow semantics:

$$\mathfrak{M}, w \Vdash \varphi \mathrel{-\!\!\!3} \psi \quad \text{iff} \quad \text{for all } v \in W \text{ (if } w \preceq v \text{ and } R[v] \subseteq V(\varphi) \text{ then } R[v] \subseteq V(\psi))$$

Example:

$$\begin{tikzcd}
v \arrow[r, squiggly] \arrow[d, bend left] & s \Vdash p \\
w \arrow[r, squiggly] \arrow[u, bend left] & u \Vdash q
\end{tikzcd}
\qquad w \nVdash p \mathrel{-\!\!\!3} r \qquad (R[w] \subseteq V(p), \quad R[w] \not\subseteq V(r))$$

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$w \nVdash p \strictif r \qquad (R[w] \subseteq V(p), \quad R[w] \nsubseteq V(r))$

$w, v \nVdash \Box p \qquad$ (because $u \nVdash p$)

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\end{tikzcd}$$

$$\begin{array}{ll}
w \nVdash p \strictif r & (R[w] \subseteq V(p), \quad R[w] \nsubseteq V(r)) \\
w, v \nVdash \Box p & (\text{because } u \nVdash p) \\
w \Vdash \Box p \to \Box r &
\end{array}$$

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$w \nVdash p \mathrel{-\!\!\!3} r \qquad (R[w] \subseteq V(p), \quad R[w] \not\subseteq V(r))$

$w, v \nVdash \Box p \qquad (\text{because } u \nVdash p)$

$w \Vdash \Box p \to \Box r$

So $(\Box p \to \Box r) \to (p \mathrel{-\!\!\!3} r)$ is not valid in flat semantics

Upward flat semantics

An upward flat model is a tuple $(W, \preceq, R, V)$ such that

- $(W, \preceq)$ is a preordered set
- R is a relation on W such that $wRv \leq u$ implies wRu
- $V : \mathrm{Prop} \to \mathsf{up}(W, \preceq)$ is a valuation

Arrow semantics:

$$\mathfrak{M}, w \Vdash \varphi \multimap \psi \quad \text{iff} \quad \text{for all } v \in W \text{ (if } w \preceq v \text{ and } R[v] \subseteq V(\varphi) \text{ then } R[v] \subseteq V(\psi))$$

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Arrow semantics:

$$\mathfrak{M}, w \Vdash \varphi \multimapinv \psi \quad \text{iff} \quad \text{for all } v \in W \text{ (if } w \preceq v \text{ and } R[v] \subseteq V(\varphi) \text{ then } R[v] \subseteq V(\psi))$$

Easier correspondence:

- A flat frame validates (4_a) $\varphi \multimapinv \Box\varphi$ iff

$$\begin{tikzcd}
 & u \arrow[r, squiggly] & s \\
w \arrow[r, squiggly] \arrow[rr, squiggly, bend right] & v \arrow[u] & t \arrow[u, dashed]
\end{tikzcd}$$

- An upward flat frame validates (4_a) iff R is transitive

Lemma

If a flat frame $\mathfrak{F} = (W, \preceq, R)$ *is pointwise downward directed, then it validates* (di)

Definition

Let $\mathfrak{M} = (W, \leq, R, V)$ be a sharp model and $\bullet$ a symbol such that $\bullet \notin W$. Define $\mathfrak{M}^\flat = (W^\flat, \preceq, \mathcal{R}, V^\flat)$, where

$$W^\flat := \{(w, v) \mid w \in W \text{ and } wRv\} \cup \{(w, \bullet) \mid w \in W \text{ and } R[w] = \emptyset\} \qquad (w, v) \preceq (w', v') \quad \text{iff} \quad w \leq w'$$

$$V^\flat(p) := \{(w, v) \in W^\flat \mid w \in V(p)\} \qquad (w, v)\mathcal{R}(w', v') \quad \text{iff} \quad v \leq w'$$

Theorem

Let $\mathfrak{M} = (W, \leq, R, V)$ *be a sharp model, Then for all* $(w, v) \in W^\flat$ *and all formulas* φ*, we have* $\mathfrak{M}, w \Vdash^\sharp \varphi$ *if and only if* $\mathfrak{M}^\flat, (w, v) \Vdash \varphi$.

Theorem

The logic HLC$^\sharp$ *is sound and complete with respect to the class of pointwise downward directed flat frames.*

Canonical model construction?

Theory: deductively closed set of formulas that does not contain $\bot$

Prime theory: theory Γ such that $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$

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Intuitionistic Kripke frame for $\Box$: (Prime theories, $\subseteq$)

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Intuitionistic Kripke frame for $\Box$: (Prime theories, $\subseteq$)

Modal relation R: $\Gamma R \Delta$ iff $\top \strictif \varphi \in \Gamma$ implies $\varphi \in \Delta$

Does not work:

- Not enough points
- We crucially rely on the intuitionistic preorder

Canonical model construction á la CK

Theory: deductively closed set of formulas that does not contain $\bot$

Prime theory: theory Γ such that $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$

Canonical model construction á la CK

Theory: deductively closed set of formulas that does not contain $\perp$

Prime theory: theory Γ such that $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$

Segment: (Γ, U) where $\Gamma \in$ Prime theories and $U \subseteq$ Prime theories, such that

- if $\varphi \prec\!\!\!\!- \psi \in \Gamma$ and $\varphi \in \Delta$ for all $\Delta \in U$,
  then $\psi \in \Delta$ for all $\Delta \in U$

Canonical model construction á la CK

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Segment: (Γ, U) where $\Gamma \in$ Prime theories and $U \subseteq$ Prime theories, such that

- if $\varphi \,\text{-}\!\!\!\ni\, \psi \in \Gamma$ and $\varphi \in \Delta$ for all $\Delta \in U$,
  then $\psi \in \Delta$ for all $\Delta \in U$

Canonical model: $(\text{Segments}, \precsim, R, V)$ where $V(p) = \{(\Gamma, U) \mid p \in \Gamma\}$ and

$$(\Gamma, U) \precsim (\Gamma', U') \quad \text{iff} \quad \Gamma \subseteq \Gamma', \qquad (\Gamma, U) R (\Gamma', U') \quad \text{iff} \quad \Gamma' \in U$$

Theorem

$\mathsf{HLC}^\flat$ *and* $\mathsf{HLC}^\flat \oplus (\mathsf{s}_\Box)$ *are strongly complete with respect to the class of (upward-)flat frames*

Theorem

$\mathsf{HLC}^\flat$ *and* $\mathsf{HLC}^\flat \oplus (\mathsf{s}_\Box)$ *have the finite model property*

Excluded middle

$$p \vee \neg p$$

- Corresponds to $\preceq$ being symmetric
- No collapse to (classical) Kripke semantics (recall the example?)
- Opportunity to use our semantics for completeness results for subsystems of standard interpretability logics such as ILM and ILP

Theorem

Let $\mathrm{Ax} \subseteq \{(\mathsf{s}_\Box), (\mathsf{em})\}$*, then* $\mathsf{HLC}^\flat \oplus \mathrm{Ax}$ *is strongly complete w.r.t. (upward-)flat frames and has FMP*

Transitivity, two ways

		(upward-)flat correspondent	sharp correspondent
(4_a)	$p \strictif \Box p$	R is transitive	R is gathering
(p_a)	$(p \strictif q) \to \Box(p \strictif q)$	something complicated	R is transitive

Gathering: $\begin{tikzcd} w \arrow[r, squiggly, -{Circle}] & v \arrow[r, squiggly, bend left, -{Circle}] \arrow[r, dashed, bend right] & u \end{tikzcd}$

With the same canonical model construction we get:

Theorem

Let $\mathrm{Ax} \subseteq \{(\mathsf{s}_\Box), (\mathsf{em}), (\mathsf{p_a})\}$, *then* $\mathsf{HLC}^\flat \oplus \mathrm{Ax}$ *is strongly complete w.r.t. (upward-)flat frames and has FMP*

But not for (4_a)! We may have $(\Gamma, U)R(\Gamma', U')R(\Gamma'', U'')$ while $\Gamma'' \notin U$.

Using a "pruned" version of the canonical model construction we get:

Theorem

Let $\mathrm{Ax} \subseteq \{(\mathsf{s}_\Box), (\mathsf{em}), (4_a)\}$, *then* $\mathsf{HLC}^\flat \oplus \mathrm{Ax}$ *is strongly complete w.r.t. (upward-)flat frames*

S4 our way

(4_a) $p \strictif \Box p$ $\qquad$ $(t_\Box)$ $\Box p \to p$

Then

$$\Box p \leftrightarrow \Box\Box p$$

and using the pruned canonical model we can prove:

Theorem

$\mathsf{HLC}^\flat \oplus (4_a) \oplus (t_\Box)$ *is strongly complete w.r.t. (upward-)flat frames and has the FMP*

Future flat work

Extensions

- More extensions, especially those related to interpretability logics
- Sahlqvist-style correspondence and canonicity

Additional flat modalities

- A diamond modality (as in CK)
- A binary dual of $\prec$ and other related connectives
- Definability over "cartesian" frames

Gödel-McKinsey-Tarski translation in the flat setting

- There is an obvious translation into $\mathsf{S4} \otimes \mathsf{K}$
- The descriptive frames of both logics seem to differ too much for a Blok-Esakia style theorem

More todos

- Developing further IELE
- Curry-Howard, especially in the flat setting
- Adapting existing formalizations (mostly Rocq in Birmingham)
- Developing further the dialgebraic perspective ...

Bonus slides

That pro-Brouwer quote from Lewis

[T]he mathematical logician Brouwer has maintained that the law of the Excluded Middle is not a valid principle at all. The issues of so difficult a question could not be discussed here; but let us suggest a point of view at least something like his. ... The law of the Excluded Middle is not writ in the heavens: it but reflects our rather stubborn adherence to the simplest of all possible modes of division, and our predominant interest in concrete objects as opposed to abstract concepts. The reasons for the choice of our logical categories are not themselves reasons of logic any more than the reasons for choosing Cartesian, as against polar or Gaussian coördinates, are themselves principles of mathematics, or the reason for the radix 10 is of the essence of number.

"Alternative Systems of Logic", *The Monist*, 1932

Collapsing $\strictif$ to $\Box$, sharply and classically

On the one hand,

$$\varphi \mathrel{-\!\!\mathsf{3}} \psi \leq \varphi \mathrel{-\!\!\mathsf{3}} (\varphi \to \psi)$$

On the other hand,

$$\begin{aligned} \top &= (\neg\varphi) \to (\varphi \to \psi) \\ &= (\neg\varphi) \mathrel{-\!\!\mathsf{3}} (\varphi \to \psi) \end{aligned}$$

Putting both hands together

$$\varphi \mathrel{-\!\!\mathsf{3}} \psi \leq (\varphi \mathrel{-\!\!\mathsf{3}} (\varphi \to \psi)) \wedge ((\neg\varphi) \mathrel{-\!\!\mathsf{3}} (\varphi \to \psi))$$

Now using (di)

$$\varphi \mathrel{-\!\!\mathsf{3}} \psi \leq (\varphi \vee \neg\varphi) \mathrel{-\!\!\mathsf{3}} (\varphi \to \psi)$$

In the opposite direction,

$$\alpha \strictif (\beta \to \gamma) \leq (\alpha \wedge \beta) \to \gamma$$

is another variant of $(\mathsf{k_a})$. Thus

$$(\varphi \vee \neg\varphi) \strictif (\varphi \to \psi) \leq ((\varphi \vee \neg\varphi) \wedge \varphi) \strictif \psi \qquad = \varphi \strictif \psi$$

Putting both slides together

$$\varphi \strictif \psi = (\varphi \vee \neg\varphi) \strictif (\varphi \to \psi)$$

Note no other classical tautology would work as the strict antecedent!

Provability and preservativity in Heyting Arithmetic

Schematic logics: a framework for arithmetical interpretations

- Extend $\mathcal{L}_{\mathsf{IPC}}$ to $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$ with operators $\circledcirc_0, \ldots, \circledcirc_k$

  where $\circledcirc_i$ has arity n_i
- F assigns to every $\circledcirc_i$ an arithmetical formula $A(v_0, \ldots, v_{n_i-1})$

  where all free variables are among the variables shown
- We write $\circledcirc_{i,F}(B_0, \ldots, B_{n_i-1})$ for $F(\circledcirc_i)(\underline{\ulcorner B_0 \urcorner}, \ldots, \underline{\ulcorner B_{n_i-1} \urcorner})$

  Here $\underline{\ulcorner C \urcorner}$ is the numeral of the Gödel number of C
- f maps $Vars$ to arithmetical sentences. Define $(\varphi)^f_F$:
  - $(p)^f_F := f(p)$
  - $(\cdot)^f_F$ commutes with the propositional connectives
  - $(\circledcirc_i(\varphi_0, \ldots, \varphi_{n_i-1}))^f_F := \circledcirc_F((\varphi_0)^f_F, \ldots, (\varphi_{n_i-1})^f_F)$

- Let T be an arithmetical theory

  An extension of i-EA, the intuitionistic version of Elementary Arithmetic, in the arithmetical language
- A modal formula in $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$ is T-valid w.r.t. F iff,
  for all assignments f of arithmetical sentences to $Vars$,
  we have $T \vdash (\varphi)^f_F$.
- Write $\Lambda_{T,F}$ for the set of $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$-formulas that are T-valid w.r.t. F.
- Of course, $\Lambda_{T,F}$ interesting only for well-chosen F

- Consider first empty set of $\circledcirc$'s
- …i.e., our language is pure $\mathcal{L}_{\mathsf{IPC}}$
- If T is Heyting Arithmetic (HA), then Λ_T is precisely IPC
- The property of theories that $\Lambda_T = \mathsf{IPC}$ is called *the de Jongh property*.
- There are theories for which Λ_T is an intermediate logic strictly between IPC and CPC.
- In fact, if you pick any intermediate logic Θ with the fmp, just extend HA with the corresponding scheme

- Now consider a single unary $\circledcirc = \Box$ and any arithmetical theory T ...
- ... which comes equipped with a $\Delta_0(\mathsf{exp})$-predicate α_T encoding its axiom set.
- Let provability in T be arithmetised by prov_T.
- Set $\mathsf{F}_{0,T}(\Box) := \mathsf{prov_T}(v_0)$. Let $\Lambda^*_T := \Lambda_{T,\mathsf{F}_{0,T}}$.
- Intuitionistic Löb's logic i-GL is given by the following axioms over IPC.

  N $\vdash \varphi \ \Rightarrow \vdash \Box\varphi$
  K $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  L $\vdash \Box(\Box\varphi \to \varphi) \to \Box\varphi$

The theory GL is obtained by extending $\mathsf{i\text{-}GL}$ with classical logic

If T is a Σ^1_0-sound classical theory, then $\Lambda^*_T = \mathsf{GL}$ (Solovay)

In contrast, the logic $\mathsf{i\text{-}GL}$ is not complete for HA. For example, the following principles are valid for HA but not derivable in $\mathsf{i\text{-}GL}$.

- $\vdash \Box\neg\neg\Box\varphi \to \Box\Box\varphi$.
- $\vdash \Box(\neg\neg\Box\varphi \to \Box\varphi) \to \Box\Box\varphi$
- $\vdash \Box(\varphi \vee \psi) \to \Box(\varphi \vee \Box\psi)$.

Still unknown what the ultimate axiomatization is

- Many possible interpretations of a binary connective

  not all of them validating Lewis-Heyting axioms!
- Interpretability
- Π^0_1-conservativity
- Σ^0_1-preservativity

  classically, the last two intertranslatable, like $\Box$ and $\Diamond$

- The notion of Σ^0_1-preservativity for a theory T (Visser 1985) is defined as follows:
- $A \strictif_T B$ iff, for all Σ^0_1-sentences S, if $T \vdash S \to A$, then $T \vdash S \to B$
- This actually validates Lewis-Heyting axioms!
- And more curious axioms are valid under this interpretation too ...

Examples of valid principles

- All theorems of $\mathsf{HLC}^\flat$

Di $(\varphi \multimapinv \chi \wedge \psi \multimapinv \chi) \to (\varphi \vee \psi) \multimapinv \chi$

"provable closure under q-realizabilty": holds for the Heyting Arithmetic or Markov Arithmetic, but not for all extensions

4_a $\varphi \multimapinv \Box\varphi$

M_a $(\varphi \multimapinv \psi) \to (\Box\chi \to \varphi) \multimapinv (\Box\chi \to \psi)$