Decision problems for ordered monoids

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Based on joint work with Almudena Colacito, Jan Draisma, Nikolaos Galatos, and Simon Santschi

TACL 2026, 27-31 July 2026, Kraków

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$\mathcal{L}$**ogic**: *semantics of non-classical logics*.

A Sample Problem

Is there an order $\leq$ on the free abelian group $\mathbf{Z}^2$ satisfying

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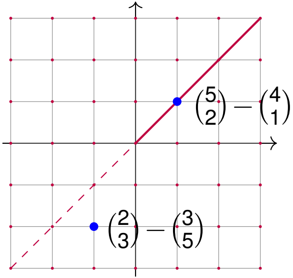

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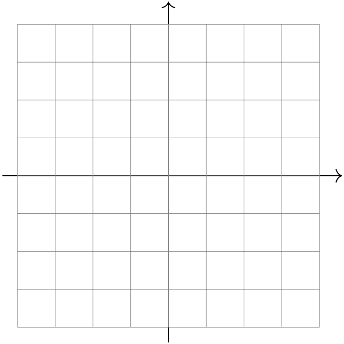

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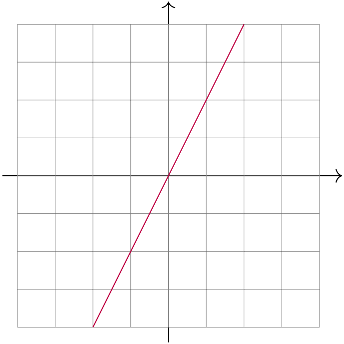

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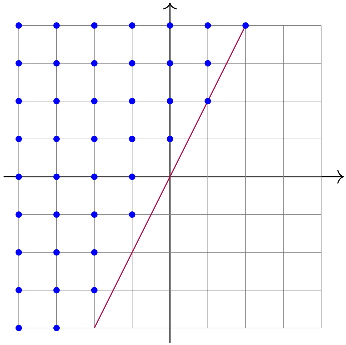

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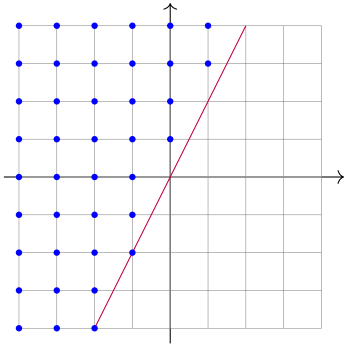

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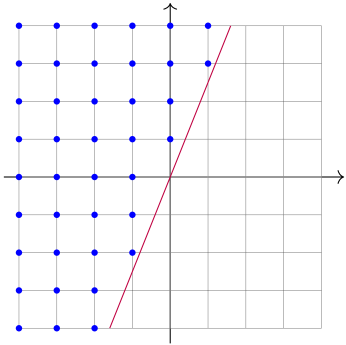

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Checking whether $\mathbf{Z}^k$ admits a order satisfying a given finite set of inequalities is in P (Khachiyan 1979), whereas the equational theory of ordered abelian groups is coNP-complete (Weispfenning 1986).

Our Topic

What can we say about orders on (commutative) monoids?

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and we can identify its left-orders with its subsemigroups P satisfying

$$M \backslash \{\mathrm{e}\} = P \cup P^{-1}.$$

The Space of Left-Orders on a Monoid

We equip the set $\mathcal{LO}(\mathbf{M})$ of left-orders on a monoid $\mathbf{M}$ with the coarsest topology such that $\{\leq \in \mathcal{LO}(\mathbf{M}) \mid a < b\}$ is open for any $a, b \in M$.

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Remark

$\mathcal{LO}(\mathbf{M})$ is a Cantor space for a countable left-orderable monoid $\mathbf{M}$ if, and only if, it has no **isolated points**: left-orders on $\mathbf{M}$ that uniquely satisfy some finite set of inequalities.

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- Every group admits finitely or uncountably many left-orders.
- E.g., the fundamental group of the Klein bottle admits precisely four left-orders, but is not orderable.

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(II) Show that (the positive cone of) a left-order on $\mathbf{F}(k)$ cannot be a finitely generated semigroup. $\square$

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This is a 2-truncated left-order on $\mathbf{F}(2)$, so – by Clay and Smith's Lemma – there exists a left-order on $\mathbf{F}(2)$ containing S.

But now...

What does this have to do with deciding equations ?

Lattice-Ordered Groups

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(i) $\langle L, \cdot, {}^{-1}, \mathrm{e} \rangle$ is a (torsion-free) group;

(ii) $\langle L, \wedge, \vee \rangle$ is a (distributive) lattice with $a \le b \;:\Longleftrightarrow\; a \vee b = b$;

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Holland's theorem can be used to prove that a group is left-orderable if, and only if, it embeds into an ℓ-group (Hollister 1965), and that the variety of ℓ-groups LG is generated by $\textbf{Aut}(\langle \mathbb{R}, \leq \rangle)$ (Holland 1976).

Another Theorem of Alternatives

Theorem (Colacito & M. 2019)

The following are equivalent for any $t_1, \ldots, t_n \in \mathrm{F}(k)$:

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(2) $\Rightarrow$ (1) Via a general proof system for left-orders on groups. $\square$

The Equational Theory of ℓ-Groups

Corollary (Holland & McCleary 1979)

The equational theory of ℓ-groups is decidable.

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Remark

Deciding equations in ℓ-groups is coNP-complete (Galatos & M. 2016), and checking whether $\mathbf{F}(k)$ admits a left-order satisfying a given finite set of inequalities is NP-complete (M. & Santschi 2025).

Another Question

What can we say about left-orders on free monoids?

Free Monoids

Let $\mathbf{FM}(k)$ denote the **free monoid** on k generators, understood as a submonoid of the free group $\mathbf{F}(k)$.

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Remark

The key ingredient here is a proof that an inverse-free equation is valid in all ℓ-groups if, and only if, it is valid in all distributive ℓ-monoids.

Left-Orders on Free Monoids

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Open Problem

Is $\mathcal{LO}(\mathbf{FM}(k))$ a Cantor space for each $k \geq 2$?

And Another Question

What can we say about orders on free monoids and groups?

Orders on Free Groups and Monoids

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Open Problems

Can we determine if there is an order on a free group (resp. monoid) satisfying a finite set of inequalities? Is the equational theory of ordered groups (resp. monoids) decidable?

A Final Question

What can we say about orders on free commutative monoids ?

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The **free commutative monoid** on k generators is $\mathbf{N}^k$.

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so orders on $\mathbf{N}^k$ are in one-to-one correspondence with orders on $\mathbf{Z}^k$, and checking if $\mathbf{N}^k$ admits an order satisfying a finite set of inequalities corresponds again to deciding equations in ordered abelian groups.

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This reflects the fact that $\mathbf{N}^3$ admits a preorder but no order satisfying

$$\begin{pmatrix}5\\0\\0\end{pmatrix} < \begin{pmatrix}2\\1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\3\end{pmatrix} < \begin{pmatrix}2\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\3\\0\end{pmatrix} < \begin{pmatrix}2\\1\\1\end{pmatrix}.$$

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Indeed, deciding universal sentences in ordered commutative monoids corresponds to determining whether $\mathbf{N}^k$ admits a preorder satisfying a finite boolean combination of inequalities.

Preorders on Commutative Monoids

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Theorem (Robbiano 1985)

For any preorder $\leq$ on $\mathbf{Z}^k$, there exists a homomorphism $\rho\colon \mathbf{Z}^k \to \mathbf{R}^k$ such that $u \leq v \iff \rho(u) \leq_{lex} \rho(v)$.

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Example

The set of left-orders on **N** is $\mathbb{Z} \cup \{-\infty, \infty\}$ where $i \in \mathbb{N}$ denotes

$$0 < 1 < \cdots < i \approx i+1 \approx i+2 \approx \cdots,$$

∞ is the standard order, and $-i$ is the dual of $i \in \mathbb{N} \cup \{\infty\}$.

The Space of Preorders on a Commutative Monoid

We equip the set $\mathcal{PO}(\mathbf{M})$ of preorders on $\mathbf{M}$ with the coarsest topology such that $\{\leq \in \mathcal{PO}(\mathbf{M}) \mid a < b\}$ is closed for any $a, b \in M$.

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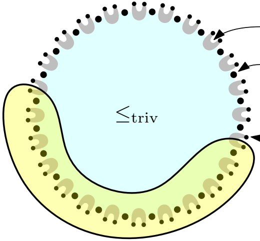
E.g., $\mathcal{PO}(\mathbf{Z}^2)$ looks something like...

$(1,2) < (0,0)$



$(u_1, u_2) \leq (v_1, v_2) \Leftrightarrow$

$u_1 + u_2 \leq v_1 + v_2$

$\sqrt{3} \cdot u_1 + u_2 \leq \sqrt{3} \cdot v_1 + v_2$

$Au \leq_{\text{lex}} Av$ with $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

Main Results

Theorem (Draisma, M., Santschi 2026)

For each $k \in \mathbb{N}$, determining whether there is a preorder on $\mathbf{N}^k$ that satisfies a finite boolean combination of inequalities is decidable.

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One procedure seeks a suitable 'd-truncated' preorder on $\mathbf{N}^k$ for $d = 1, 2, \ldots$ outputting 'fail' if none exists;

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Theorem (Draisma, M., Santschi 2026)

The universal theory of ordered commutative monoids is decidable.

Concluding Remarks

- Spaces of (pre)orders on monoids provide a framework for investigating related decision problems.

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- Spaces of (pre)orders on monoids provide a framework for investigating related decision problems.
- Deciding equations in ℓ-groups is related to checking if there is a left-order on a free group satisfying finitely many inequalities.
- Deciding universal sentences in ordered commutative monoids is related to checking if there is a preorder on a free commutative monoid satisfying a finite boolean combination of inequalities.
- Intriguing open problems include decidability of the equational theories of ordered groups and monoids, corresponding to the existence of certain orders on free groups and monoids.

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