Relations in Order-enriched Categories

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Relations

Relations that are compatible with partial order on structures

A weakening relation between poset is $R \subseteq X \times Y$ s.t.

$$\frac{x \Rightarrow x' \qquad x' \mathrel{R} y' \qquad y' \Rightarrow y}{x \mathrel{R} y}\,[W]$$

Here $\Rightarrow$ for emphasis. More generally, it is a partial order.

Many examples

- Well-inside: $a \lessdot b$ iff $\exists c.1 \leq b \vee c$ and $c \wedge a = 0$.
- Rather below: $a \prec b$ iff $b \leq \bigvee I$ implies $a \in I$ for all directed downsets I.
- $D \leq {\downarrow} a$ – relating (say) downsets to elements
- $a \in D$ – relating elements to downsets.
- "Distributively below": $a \sqsubseteq b$ iff $\exists w.a \leq b \vee w \& w \wedge a \leq b$

Categories of weakening relations

In many categories, we can make sense of $\mathrm{wRel}(\mathcal{X})$

- Objects are same as $\mathcal{X}$;
- Morphisms are subobjects of $X \times Y$ that are also weakening relations.
- Composition is the usual relation composition;
- 1_X is the order $\leq_X$.

Examples

- Posets: All weakening relations.
- Semilattices: $a \mathrel{R} b$ and $a \mathrel{R} c$ implies $a \mathrel{R} b \wedge c$.
- Lattices: also, $a \mathrel{R} c$ and $b \mathrel{R} c$ implies $a \vee b \mathrel{R} c$.
- Frames: also $\{a_i \mathrel{R} c\}_i$ implies $\bigvee_i a_i \mathrel{R} c$.
- Quantales, etc.

Where does this lead?

Some clues

The connection between **Set** and **Rel** is well-charted.

- **Set** to **Rel**: take "relation" to mean subobject of $X \times Y$.
  Set has enough structure to allow for composition of relations.
- **Rel** to **Set**: Take a "map" to be an adjoint pair of relations
  $\Delta_X \subseteq R\,;S$ and $S\,;R \subseteq \Delta_Y$ – this makes sense because relations are ordered.
- But functions (morphisms in **Set**) are discretely ordered.

Pos and **wRel**(**Pos**) – and others – are related in (almost) the same way.

- Adjoint pairs in wRel($\mathcal{X}$) determine homomorphisms.
- A weakening relation R is represented by a special sort of subobject of $X \times Y$.

Order-enriched Categories

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We assume for all categories and functors:

- The morphisms $\mathcal{A}(X, Y)$ form a poset.
- Composition preserves order.
- Functors preserve order.

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Limits have more structure, and there are more kinds of limits

- Product $X \times Y$ now has to satisfy $\mathcal{A}(W, X \times Y) \simeq \mathcal{A}(X, Y) \times \mathcal{A}(W, Y)$ as posets.
- All finite limits are constructed from terminal, pullbacks and $2 * X$:

$$\begin{tikzcd} & 2 * X \arrow[dl, "p_X"'] \arrow[dr, "q_X"] & \\ X & \leq & X. \end{tikzcd}$$ universally.

Adjoint pairs

- In any (order-enriched) category, pairs of morphisms $(p{:}h \to iXY, \psi{:}Y \to X)$ may be adjoint:

$$1_X \leq \psi\varphi \qquad \text{and} \varphi\psi \leq 1_Y.$$

- I suggestively write $f = (\hat{f}, \check{f})$ for an adjoint pair.
- Adjoint pairs form a category:
  - Identity is $(1_X, 1_X)$
  - Composition is $g \circ f = (\hat{g}\hat{f}, \check{f}\check{g})$.
  - Pairs are ordered by the left adjoint: $f \leq g$ iff $checkf \leq \check{g}$.
- Map($\mathcal{X}$) is the category of adjoint pairs.

An Easy Step

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Question
What structure on $\mathcal{X}$ is preserved in Map($\mathcal{X}$)? – From examples, seems not much.

Another (Less) Easy Part in the Other Direction

Abstract Relations

- If a category has (binary) products, we can define a relation to be a subobject of $A \times B$.
- We might want special subobjects, but this is the basic idea.
- More simply, take (iso classes of) jointly order-embedding spans $A \xleftarrow{p} R \xrightarrow{q} B$.

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But how to we know that abstract relations form a category?

- What is composition?
- Is composition associative?
- What are the identity relations?

Order Regular Categories

Order-regular category (Kurz & Velebil)
A category $\mathcal{A}$ is order-regular iff

- $\mathcal{A}$ has all finite weighted limits
- $\mathcal{A}$ has strong/order-embedding factorization
- strong morphisms are stable under pullbacks

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Idea (for those familiar with ordinary regular categories)
Order-regularity really is regularity suitably adapted to order-enrichment:

- All finite limits means *all* finite limits including weighted ones
- Subobjects are defined by order-embeddings; $mf \leq mg$ implies $f \leq g$.
- Strong morphisms are defined with respect to these subobjects.

Both Sorts of Categories Require (Not Necessarily Cartesian) Products

Symmetric Monoidal Categories
We need "products". So

- A "multiplication" functor $\otimes : \mathcal{X}^2 \to \mathcal{X}$
- A "unit" object $\mathbb{I}$
- Natural isomorphisms

$$a : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$$

$$r : X \otimes \mathbb{I} \simeq X$$

$$l : \mathbb{I} \otimes X \simeq X$$

$$c : X \otimes Y \simeq Y \otimes X$$

$$x : (W \otimes X) \otimes (Y \otimes Z) \simeq (W \otimes Y) \otimes (X \otimes Z)$$ Redundant: built from a and c.

- Don't forget: The functor $\otimes$ preserves order on morphisms.

Relation composition

Relation composition

$$\begin{tikzcd}
 & & R \arrow[dl, "r"'] \arrow[dr, "s"] & & \\
 & \varphi \arrow[dl, "p_\varphi"'] \arrow[dr, "q_\varphi"] & \leq & \psi \arrow[dl, "p_\psi"'] \arrow[dr, "q_\psi"] & \\
A & & B & & C
\end{tikzcd}$$

But this might not be jointly order-embedding.

Concretely, think of $R = \{(a, b, b', c) \mid a\varphi b, b \leq b', b'\psi c\}$.

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$$
\begin{tikzcd}
 & & \varphi \fatsemi \psi \arrow[ddll, bend right, "p_{\varphi \fatsemi \psi}"'] \arrow[ddrr, bend left, "q_{\varphi \fatsemi \psi}"] & & \\
 & & R \arrow[u, two heads] \arrow[dl, "r"'] \arrow[dr, "s"] & & \\
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So we need image factorization.

Concretely, projects as $\varphi \fatsemi \psi = \{(a, c) \mid \exists b, b'.(a, b, b', c) \in R\}$.

What is Needed to Make Sense of $\varphi \fatsemi \psi$

Order pullbacks

Given $\varphi \xrightarrow{q_\varphi} R \xleftarrow{p_\psi} \psi$ an order pullback is

$$
\begin{tikzcd}
 & R \arrow[dl, "r"'] \arrow[dr, "s"] & \\
\varphi \arrow[dr, "q_\varphi"'] \arrow[rr, phantom, "\leq"] & & \psi \arrow[dl, "p_\psi"] \\
 & B &
\end{tikzcd}
$$

that is universal. These are also known as commas.

Factorization
Every $f{:}X \to Y$ factors as $f = me$ where

- m is an order-embedding: $mh \leq mk$ implies $h \leq k$, and
- e is a strong morphism: for any $$\begin{tikzcd} W \arrow[r, "e"] \arrow[d, "u"] & X \arrow[d, "v"] \\ Y \arrow[r, hook, "m"] & Z \end{tikzcd}$$ with m order-embedding, there

  is a (unique) diagonal d so that $$\begin{tikzcd} W \arrow[r, "e"] \arrow[d, "u"] & X \arrow[d, "v"] \arrow[dl, "d"'] \\ Y \arrow[r, hook, "m"] & Z. \end{tikzcd}$$

One More Ingredient to Make Sense of $\varphi \fatsemi \psi$

With order pullbacks and image factorization

$$
\begin{tikzcd}
 & & \varphi \fatsemi \psi \arrow[ddd, ll, bend right, "p_{\varphi \fatsemi \psi}"'] \arrow[dddrr, bend left, "q_{\varphi \fatsemi \psi}"] & & \\
 & & R \arrow[u, two heads] \arrow[dl, "r"] \arrow[dr, "s"'] & & \\
 & \varphi \arrow[dl, "p_\varphi"'] \arrow[dr, "q_\varphi"] & \leq & \psi \arrow[dl, "p_\psi"'] \arrow[dr, "q_\psi"] & \\
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makes sense.

But do we have a category?

- Composition is associative exactly when strong morphisms (images) are stable under pullbacks – precisely, Kurz&Velebil's order-regularity.
- But $\fatsemi$ does not have identities with respect to *all* jointly order-embeddings.

Order-Regular Categories and their Relation Categories

Reminder
A category is order-regular (pace Kurz and Velebil) if

- It has all finite weighted limits (a terminal object, pullbacks and $2 * X$).
- every $f{:}X \to Y$ factors into an strong then order-embedding;
- Strong morphisms are stable under pullbacks.

Order-Regular Categories and their Relation Categories

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The categories $\mathrm{wRel}(\mathcal{A})$
For any order regular category $\mathcal{A}$ define

- $\mathbf{1}_X = X \overset{p_X}{\leftarrow} 2 * X \overset{q_X}{\rightarrow} X$.
- $\mathrm{wRel}(\mathcal{A})$: category of iso classes of jointly order-embedding spans where

$$\mathbf{1}_X \mathbin{;} \varphi \mathbin{;} \mathbf{1}_Y \leq \varphi$$

.

Back the Other Way

From “relation” morphisms to maps

- Recall that $\mathrm{Map}(\mathcal{X})$ always gives us a category.
- Since we suppose functors preserve order, $\mathrm{Map}(-)$ extends to functors.
- Map also preserves natural isomorphisms between functors.
- Map is, at least, a 2-functor on the 2-category of order-enriched categories, functors between them, and *natural isomorphisms* between functors.

A Useful Theorem

- For any symmetric monoidal category $\mathcal{X}$, $\mathrm{Map}(\mathcal{X})$ is also symmetric monoidal.

The big question

- When is $\mathrm{Map}(\mathcal{X})$ a Kurz-Velebil order-regular category?

Fun Fact

In a symmetric monoidal category

- $\mathbb{I}$ is terminal iff there is a natural transfornation

$$k : X \to \mathbb{I}$$

  satisfying $k_{\mathbb{I}} = 1_{\mathbb{I}}$.

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  satisfying $k_{\mathbb{I}} = 1_{\mathbb{I}}$.
- If $\mathbb{I}$ is terminal, then $\otimes$ is cartesian product iff there is a natural commutative comonoid wrt $\mathbb{I}$ and $\otimes$
- That is, we have a natural

  $$d : X \to X \otimes X$$

  so that k and d satisfy commutative comonoid laws.

The Cartesian Laws

Natural transformations k and d form a commutative comonoid:

$$a \circ (d \otimes 1) \circ d = (1 \otimes d) \circ d$$

$$r \circ (k \otimes 1) \circ d = 1$$

$$l \circ (1 \otimes k) \circ d = 1$$

$$x \circ (d \otimes d) \circ d = d_{(2)} \circ d \qquad \text{where } (d_{(2)})_X = d_{X \otimes X}$$

$$\begin{tikzcd}
 & X \arrow[dl, "d"'] \arrow[d, "d"] \\
X \otimes X \arrow[d, "1 \otimes d"] & X \otimes X \arrow[d, "d \otimes 1"] \\
X \otimes (X \otimes X) \arrow[r, "a"] & (X \otimes X) \otimes X
\end{tikzcd}
\quad
\begin{tikzcd}
 & X \arrow[dl, "d"] \arrow[dd, "1"] \\
X \otimes X \arrow[d, "1 \otimes k"] & \\
X \otimes \mathbb{I} \arrow[r, "r"] & X
\end{tikzcd}
\quad
\begin{tikzcd}
 & X \arrow[dl, "d"] \arrow[dd, "1"] \\
X \otimes X \arrow[d, "k \otimes 1"] & \\
\mathbb{I} \otimes X \arrow[r, "l"] & X
\end{tikzcd}
\quad
\begin{tikzcd}
 & X \arrow[dl, "d"] \arrow[d, "d"] \\
X \otimes X \arrow[d, "d \otimes d"] & X \otimes X \arrow[d, "d_{(2)}"] \\
(X \otimes X) \otimes (X \otimes X) \arrow[r, "x"] & (X \otimes X) \otimes (X \otimes X)
\end{tikzcd}$$

An alternative to naturality

Adjoint Transformations
An adjoint transformation $\nu{:}F \to G$ consists of two families of morphisms $\hat{\nu}_X{:}F(X) \to G(X)$ and $\nu\alpha_X{:}G(X) \to F(X)$ so that

$$F(\varphi) \leq \check{\nu}_Y \circ G(\varphi) \circ \hat{\nu}_X \qquad \text{and} \qquad \hat{\nu}_Y \circ F(\varphi) \circ \check{\nu}_X \leq G(\varphi)$$

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These generalize natural isomorphisms (not general natural transformations)
If $a{:}F \to G$ is a natural isomorphism, then $(a, a^{-}1)$ is an adjoint transformation.

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Adjoint transformations compose

- $\nu \circ \mu$ is defined by $\hat{\nu} \circ \hat{\mu}$ and $\check{\mu} \circ \check{\nu}$.
- Composing to identity = a pair of natural isomorphisms.
- “Vertical” composition is also well-defined.

Two 2-categories

Natural versus adjoint transformations
We have two distinct 2-categories:

- **OCat**: order-enriched categories, functors, and natural transformations.
- $\mathbf{OCat}_{\dashv}$: order-enriched categories, functors, and adjoint transformations.

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Natural versus adjoint transformations
We have two distinct 2-categories:

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- $\mathbf{OCat}_\dashv$: order-enriched categories, functors, and adjoint transformations.

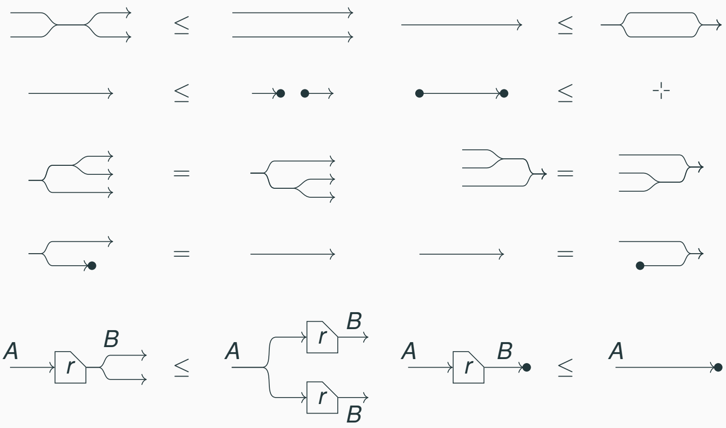
Theorem

- Map sends an adjoint transformation to a natural transformation. Thus,
- Map:$\mathbf{OCat}_\dashv \to \mathbf{OCat}$ is a 2-functor.
- Define an adjoint cartesian category to be a symmetric monoidal category with comonoid of adjoint transformations (κ, δ) so that $\kappa_{\mathbb{I}} = 1_{\mathbb{I}}$.

Cartesian bicategories
Adjoint cartesian categories are almost exactly the cartesian bicategories of Carboni & Walter – though our axiomatization is slightly different.

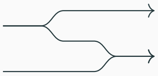
Aside: Wiring diagrams for cartesian bicategories

For adjoint cartesianness, every object is equipped with morphisms ($\hat{\delta}_A$ split; $\check{\delta}_A$ fuse; $\hat{\kappa}_A$ terminate; $\check{\kappa}_A$ initiate) that can be depicted in wiring diagrams.

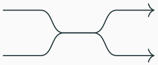


Frobenius and categories of ordinary relations

Carboni and Walter were concerned with internal relations in *ordinary* regular categories. Their cartesian bicategories satisfy this (Frobenius) axiom:



$\leq$



Fact (from Carboni&Walter)
If Frobenius holds, then $\mathrm{Rel}(\mathrm{Map}(\mathcal{X})) \simeq \mathcal{X}$ as ordinary categories (order on morphisms is forgotten).

Remarks

- Note the pattern. $\geq$ always holds (it can be read as a ”dual” distributive law).
- In a sense, Frobenius is dual to Boole.

Now We Can Start to Put Things Together

Some refined 2-categories

- **OCart**: The 2-category of symmetric monoidal categories with natural comonoid and $\mathbb{I}$ terminal, functors preserving $\otimes$ and$\mathbb{I}$, and 2-cells are natural transformations.
- **OCart**$_\dashv$: Same as **OCart** but with comonoid of adjoint transformations, and 2-cells are adjoint transformations.
- **OReg**: The 2-category of order regular categories, order regular functors (preserving finite limts and strong morphisms), natural transformations.

How these relate so far

- Map:**OCart**$_\dashv \to$ **OCart**
- wRel:**OReg** $\to$ **OCart**$_\dashv$
- **OReg** $\subseteq$ **OCart**.

At last

We need to understand the image of wRel

- Every $\varphi{:}X \to Y$ is tabulated (a right/left adjoint factorization system):

$$f = (\hat{f}, \check{f}){:}R \to X \quad \text{and} \quad g = (\hat{g}, \check{g}){:}R \to Y$$

  so that

  - $\check{f} \mathbin{;} \hat{g} = \varphi$, and
  - if $\check{h} \mathbin{;} \hat{k} \leq \varphi$, the pair (h, k) factors uniquely through (f, g).

- Every $\varphi{:}X \to X$ if $\varphi \leq 1_X$ has a core (an adjoint pair representing its diagonal part):

$$c = (\hat{c}, \check{c}){:}C \to$$

  so that

  - $\check{c} \mathbin{;} \hat{c} \leq \varphi$ and $1_C = \hat{c} \mathbin{;} \check{c}$; and
  - if $\check{s} \mathbin{;} \hat{s} \leq \varphi$ and $1_S \leq \hat{s} \mathbin{;} \check{s}$, then s factors ()uniquely) through c.

Main Theorem

Categories of weakening relations

Define **wRel** to be the sub-2-category of $\mathbf{OCart}_{\dashv}$ where all morphisms are tabulated and all $\varphi \leq 1_X$ have cores.

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Define **wRel** to be the sub-2-category of $\mathbf{OCart}_{\dashv}$ where all morphisms are tabulated and all $\varphi \leq 1_X$ have cores.

Theorem

- The image of wRel:$\mathbf{OReg} \to \mathbf{OCart}_{\dashv}$ lies in **wRel**.
- The image of Map:$\mathbf{wRel} \to \mathbf{OCart}$ lies in **OReg**.
- Map(wRel($\mathcal{A}$)) $\simeq \mathcal{A}$ naturally for order-regular $\mathcal{A}$.
- wRel(Map($\mathcal{X}$)) $\simeq \mathcal{X}$ naturally for category of weakening relations $\mathcal{X}$.

What Comes Next?

Internal logic

- Ordinary regularity is characterized by regular logic ($\exists$, &, $=$ fragment of FOL).
- Naively, one might hope order-regularity is characterized by $\exists$, &, $\leq$.
- Problems: $\exists y.\varphi(x, y) \& \psi(z, y)$ when y appears in "positive" position.

Alex Kurz, Achim Jung and I are working on:

- How to lift a duality to relations (**DLat – Priestley**).
- How to account for relations on an order-regular category as forming a double category:
  - A category with two sorts of interacting morphisms. Or,
  - A category internal to **Cat**.
- Seemingly technical questions (how to think about splitting idempotents).

Thank you