Pointfree theories of T_0 spaces

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[1]Based on joint work with Rui Prezado

Pointfree topology

A *frame* is a complete lattice satisfying the infinite distributive law:

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i).$$

Morphisms of frames preserve finite meets and arbitrary joins.

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We call $\mathbb{S}$ the *Sierpiński space*, the two-point space where exactly one point is isolated.

Proposition

$|\Omega X| = \mathbf{Top}(X, \mathbb{S})$ *for every* $X \in \mathbf{Top}$.

The classical adjunction

For a frame L, the *spectrum* $\mathrm{pt}(L)$ is the space:

- Whose points are the set $\mathbf{Frm}(L, \mathbf{2})$;
- Whose opens are the sets $\varphi(a) := \{f \in \mathrm{pt}(L) \mid f(a) = 1\}$ for $a \in L$.

Here $\mathbf{2}$ denotes the two-element frame $\{0, 1\}$ with $0 < 1$.

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Theorem

There is a dual adjunction

$$\Omega : \mathbf{Top}^{\mathrm{op}} \leftrightarrows \mathbf{Frm} : \mathrm{pt}$$

with $\mathrm{pt} \dashv \Omega$.

This adjunction establishes frames as pointfree spaces. But it is not an equivalence. Only certain spaces are captured faithfully.

The problem: non-sober spaces

The fixpoints on the **Top** side are the *sober* spaces. A T_0-space X is sober if every frame map $\Omega X \to \mathbf{2}$ is the characteristic function of some point $x \in X$.

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The fixpoints on the **Top** side are the *sober* spaces. A T_0-space X is sober if every frame map $\Omega X \to \mathbf{2}$ is the characteristic function of some point $x \in X$. In fact, assigning every point to its characteristic function gives the sober reflection $\sigma_X : X \to \mathrm{pt}\Omega X$ of a space. Logical relations between sobriety and other separation axioms:

$$\begin{tikzcd}
 & T_2 \arrow[dl] \arrow[dr] & \\
T_1 \arrow[d] & & \text{Sober} \arrow[ddl] \\
T_D \arrow[dr] & & \\
 & T_0 &
\end{tikzcd}$$

Recall: a space is T_D iff every point is locally closed.

We are missing a pointfree treatment of non-sober spaces.

The T_D approach

Banaschewski and Pultr developed an alternative duality for T_D-spaces. We present a slight variation.

$\mathbf{Frm}_{\mathcal{E}}$ is the category of frames with maps which are well-behaved with respect to exact (= join-distributive) meets.

For $L \in \mathbf{Frm}$, define $\mathrm{pt}_{\mathcal{E}}(L) := \mathbf{Frm}_{\mathcal{E}}(L, \mathbf{2})$, topologized as in the classical case.

Theorem (Banaschewski and Pultr, 2010)

There is an adjunction

$$\Omega : \mathbf{Top}_D^{\mathrm{op}} \leftrightarrows \mathbf{Frm}_{\mathcal{E}} : \mathrm{pt}_{\mathcal{E}}$$

The fixpoints in $\mathbf{Top}_D$ are exactly the T_D-spaces.

Morphisms between adjunctions

To compare dual adjunction, we need a notion of morphism of adjunctions.[2]. A morphism between adjunctions $(F \dashv G)$ and $(F' \dashv G')$ is a pair (H, L):

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[d, "H"'] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[d, "L"] \\
\mathcal{C}' \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{D}' \arrow[l, shift left, "G'"]
\end{tikzcd}$$

together with natural transformations

$$\zeta : F'H \implies LF \qquad\qquad \xi : HG \implies G'L$$

satisfying compatibility conditions. It is a *right* morphism if ζ is an isomorphism.

[2] *Formal Category Theory*, J. W. Gray (1974)

Relating sober and T_D

There is a right morphism of adjunctions from the T_D duality to the classical one:

$$\begin{tikzcd}
\mathbf{Frm}_{\mathcal{E}}^{op} \arrow[r, shift left, "\mathrm{pt}_{\mathcal{E}}"] \arrow[r, phantom, "\perp"] \arrow[d, hook] & \mathbf{Top}_D \arrow[l, shift left, "\Omega"] \arrow[d, hook] \\
\mathbf{Frm}^{op} \arrow[r, shift left, "\mathrm{pt}"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift left, "\Omega"]
\end{tikzcd}$$

The natural transformations witnessing this are:

- The identity $\Omega(X) = \Omega(X)$ for every T_D-space X;
- The subspace inclusion $\mathrm{pt}_{\mathcal{E}}(L) \subseteq \mathrm{pt}(L)$ for every frame L.

Unifying the sober and T_D approaches

We then have a duality for some nonsober spaces. We will now show to examples to illustrate how the frame language fails to be expressive enough, both when we restrict ourselves to sober and to T_D-spaces.

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We recall the following standard result in pointfree topology.

Proposition

For every space X, a subspace inclusion $Y \subseteq X$ induces a frame surjection

$$\begin{aligned} \Omega X &\to \Omega Y \\ U &\mapsto U \cap Y. \end{aligned}$$

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Let X be a sober space. Then, all surjection $\Omega X \to L$ to a spatial L come from subspaces of X.

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Example

Let X be a sober space. Then, all surjection $\Omega X \to L$ to a spatial L come from subspaces of X. On the other hand, if X is not sober, there will always be some surjection $\Omega X \to L$ not induced by any concrete subspace $Y \subseteq X$.

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How can we extend both dualities so that on the pointfree side the spatial objects do not show us too few or too many subspaces?

The Raney approach

A solution (Suarez, 2025) captures *all* T_0-spaces using a result by Raney, which states that T_0-spaces can be completely recovered from the embedding of the opens into the *saturated* (=intersections of open) sets.

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Definition

A *Raney extension* is a pair $(L, \mathcal{F})$ where L is a frame and $\mathcal{F}$ is a collection of filters of L (satisfying certain conditions). Morphisms $f : (L, \mathcal{F}) \to (M, \mathcal{G})$ are frame morphisms with

$$f^{-1}(G) \in \mathcal{F} \quad \text{for all } G \in \mathcal{G}.$$

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$$f^{-1}(G) \in \mathcal{F} \quad \text{for all } G \in \mathcal{G}.$$

This defines the category **Raney**. There is an obvious forgetful functor $\mathcal{O}_{\mathcal{R}} : \mathbf{Raney} \to \mathbf{Frm}$.

The Raney approach

For a topological space X, consider:

$$\begin{aligned} {\uparrow}^{\Omega X} : \mathcal{P}X &\to \mathsf{Filt}(\Omega X) \\ S &\mapsto \{U \in \Omega X \mid S \subseteq U\}. \end{aligned}$$

This sends each subset $S \subseteq X$ to its filter of open neighbourhoods.
Define $\Omega_{\mathcal{R}} X := (\Omega X, {\uparrow}^{\Omega X}[\mathcal{P}X])$.

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Lemma

For every space X, $\Omega_{\mathcal{R}} X$ is a Raney extension.

This construction is functorial: $\Omega_{\mathcal{R}} : \mathbf{Top}^{\mathsf{op}} \to \mathbf{Raney}$ and gives our refinement of the "open set" functor.

The extra data (filters) remembers what opens each subset of a space X is included in.

The Raney approach

We introduce the Raney extension $\mathbf{2}_{\mathcal{R}} = (\mathbf{2}, \mathbf{2})$: the two-element frame with all its filters.

Define $\mathrm{pt}_{\mathcal{R}}(L, \mathcal{F})$ as the space:

- Whose points are: $\mathbf{Raney}((L, \mathcal{F}), \mathbf{2}_{\mathcal{R}})$;
- Whose opens are: $\{f \mid \mathcal{O}_{\mathcal{R}} f(a) = 1\}$ for $a \in L$.

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Theorem

There is an adjunction $\mathcal{O}_{\mathcal{R}} : \mathbf{Top}^{\mathrm{op}} \leftrightarrows \mathbf{Raney} : \mathrm{pt}_{\mathcal{R}}$, whose fixpoints in **Top** *are all T_0-spaces.*

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It suffices to restrict the natural adjunctions to suitable full subcategories to recover dualities for sober and T_D spaces.

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Corollary

For every T_0-space X, the functor $\Omega_{\mathcal{R}}$ determines a bijection between subspaces of X and surjections $\Omega_{\mathcal{R}} X \to R$ in **Raney** *with R spatial.*

Relating Raney to sober

There is a right map of adjunctions:

$$\begin{tikzcd} \mathbf{Raney} \arrow[r, shift left, "\mathrm{pt}_{\mathcal{R}}"] \arrow[r, phantom, "\perp"] \arrow[d, "\mathcal{O}_{\mathcal{R}}"'] & \mathbf{Top}^{\mathrm{op}} \arrow[l, shift left, "\Omega_{\mathcal{R}}"] \arrow[d, equal] \\ \mathbf{Frm} \arrow[r, shift left, "\mathrm{pt}"] \arrow[r, phantom, "\perp"] & \mathbf{Top}^{\mathrm{op}} \arrow[l, shift left, "\Omega"] \end{tikzcd}$$

The natural transformations witnessing this:

- $\mathcal{O}_{\mathcal{R}}\Omega_{\mathcal{R}}X = \Omega X$;
- $\mathrm{pt}_{\mathcal{R}}(L, \mathcal{F}) \subseteq \mathrm{pt}(L)$ is a subspace embedding.

The MT-algebra approach

Another solution (Bezhanishvili and Raviprakash, 2023) captures *all* spaces using McKinsey-Tarski algebras.

Definition

A *McKinsey-Tarski algebra* (or *MT-algebra*) is a pair (L, M) where M is a complete Boolean algebra and $L \subseteq M$ is a subframe. Elements of L are called *open* elements. Morphisms are maps of complete Boolean algebras that restrict to the subframes of open elements.

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This defines the category **MT**. When we restrict to MT-algebras that are T_0, there is a faithful forgetful functor $\mathcal{O}_{\mathcal{M}} : \mathbf{MT} \to \mathbf{Frm}$.

The MT-algebra approach

For a topological space X, consider:

$$(\Omega X, \mathcal{P}X)$$

where ΩX is the frame of opens and $\mathcal{P}X$ is the powerset Boolean algebra.

Lemma

For every space X, $(\Omega X, \mathcal{P}X)$ is an MT-algebra.

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This construction is functorial: $\mathcal{P} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{MT}$.

The extra data (the powerset) remembers all subsets of X.

The MT-algebra approach

We introduce the MT-algebra $\mathbf{2}_{\mathcal{M}} = (\mathbf{2}, \mathbf{2})$: the two-element Boolean algebra with itself as the subframe of opens.
Define $\mathrm{pt}_{\mathcal{M}}(L, M)$ as the space:

- Whose points are atoms of M (equivalently, $\mathbf{MT}((L, M), \mathbf{2}_{\mathcal{M}})$);
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whose fixpoints in **Top** *are all topological spaces.*

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There is an adjunction $\mathcal{P} : \mathbf{Top}^{\mathrm{op}} \leftrightarrows \mathbf{MT} : \mathrm{pt}_{\mathcal{M}}$, whose fixpoints in **Top** *are all topological spaces.*

Again, it suffices to restrict the natural adjunctions to suitable full subcategories to recover dualities for sober and T_D spaces.

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Theorem

*There is an adjunction $\mathcal{P} : \mathbf{Top}^{\mathrm{op}} \leftrightarrows \mathbf{MT} : \mathrm{pt}_{\mathcal{M}}$, whose fixpoints in **Top** are all topological spaces.*

Again, it suffices to restrict the natural adjunctions to suitable full subcategories to recover dualities for sober and T_D spaces.

Corollary

*For every space X, the functor $\mathcal{P}$ determines a bijection between subspaces of X and surjections $\mathcal{P}X \to M$ in **MT** with M spatial.*

Relating MT to sober

There is a right map of adjunctions:

$$\begin{tikzcd} \mathbf{MT} \arrow[r, shift left=2, "\mathrm{pt}_{\mathcal{M}}"] \arrow[r, phantom, "\perp"] \arrow[d, "\mathcal{O}_{\mathcal{M}}"'] & \mathbf{Top}^{\mathrm{op}} \arrow[l, shift left=2, "\mathcal{P}"] \arrow[d, equal] \\ \mathbf{Frm} \arrow[r, shift left=2, "\mathrm{pt}"] \arrow[r, phantom, "\perp"] & \mathbf{Top}^{\mathrm{op}} \arrow[l, shift left=2, "\Omega"] \end{tikzcd}$$

The natural transformations witnessing this:

- $\mathcal{O}_{\mathcal{M}}\mathcal{P}X = \Omega X$;
- $\mathrm{pt}_{\mathcal{M}}(L, M) \subseteq \mathrm{pt}(L)$ is a subspace embedding.

The classical adjunction as a base adjunction

We have now seen three different instances of the same phenomenon. A variation of the (Ω, pt) adjunction, possibly extending it, is connected to it by a right map of adjunctions, and in the "spectrum" direction we have a subspace inclusion as a comparison.

$$
\begin{tikzcd}
\mathbf{Frm}_{\mathcal{E}} \arrow[r, shift left, "\mathrm{pt}_{\mathcal{E}}"] \arrow[r, phantom, "\perp"] \arrow[d, hook] & \mathbf{Top_D}^{\mathrm{op}} \arrow[l, shift left, "\Omega"] \arrow[d, hook] \\
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\begin{tikzcd}
\mathbf{Raney} \arrow[r, shift left, "\mathrm{pt}_{\mathcal{R}}"] \arrow[r, phantom, "\perp"] \arrow[d, "\mathcal{O}_{\mathcal{R}}"'] & \mathbf{Top}^{\mathrm{op}} \arrow[l, shift left, "\mathcal{U}"] \arrow[d, equal] \\
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To these three, we would like to add a fourth one, essentially the archetypal example for the general theory that we will present.

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\mathbf{Top_0}^{op} \arrow[r, equal] \arrow[d, "\Omega"'] & \mathbf{Top_0}^{op} \arrow[d, equal] \\
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The natural transformations exhibiting this as a right map of adjunctions are:

- The identity $\Omega X = \Omega X$ for every T_0-space X;
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- The identity $\Omega X = \Omega X$ for every T_0-space X;
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These analogies call for a unified treatment of such natural adjunctions with $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm} : \mathrm{pt}$ as a base.

FC-categories

We now unify all these dualities in a single framework.

Definition

A **Frm**-*concrete category*, or *FC-category*, is a pair $(\mathcal{O}, \mathcal{C})$ where $\mathcal{O} : \mathcal{C} \to \mathbf{Frm}$ is faithful.

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All our examples are FC-categories:

Example

1. $\mathbf{Frm}_{\mathcal{E}} \subseteq \mathbf{Frm}$;
2. $\mathcal{O}_{\mathcal{R}} : \mathbf{Raney} \to \mathbf{Frm}$;
3. $\mathcal{O}_{\mathcal{M}} : \mathbf{MT} \to \mathbf{Frm}$;
4. $\Omega : \mathbf{Top_0}^{\mathrm{op}} \to \mathbf{Frm}$.

Each gives a different pointfree representation of spaces.

Spectrum functor: adding a point object

To define the "spectrum" functor, we only need a distinguished object.

Definition

A *pointed FC-category* (PFC) is $(\mathcal{O}, \mathcal{C}, \mathbf{2}_{\mathcal{C}})$ where $(\mathcal{O}, \mathcal{C})$ is an FC-category, $\mathbf{2}_{\mathcal{C}} \in \mathcal{C}$, and there is an isomorphism

$$\mathcal{O}\mathbf{2}_{\mathcal{C}} \cong \mathbf{2}.$$

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$$\mathcal{O}\mathbf{2}_{\mathcal{C}} \cong \mathbf{2}.$$

Given a PFC-category, define $\mathrm{pt}_{\mathcal{C}} : \mathcal{C} \to \mathbf{Top}^{\mathrm{op}}$:

- Points: $\mathcal{C}(C, \mathbf{2}_{\mathcal{C}})$;
- Opens: $\{f \mid \mathcal{O}f(a) = 1\}$ for $a \in \mathcal{O}C$.

This construction is functorial.

Cartesian lifts

We aim to make the pair $(\mathbf{2}_{\mathcal{C}}, \mathbb{S})$ into a dualizing object, in the sense of Dimov and Tholen (1989).

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To define the right adjoint $\Omega_{\mathcal{C}}$, we need to lift points from spaces.
For $C, D \in \mathcal{C}$, a *lift* of $f : \mathcal{O}C \to \mathcal{O}D$ is $\overline{f} : C \to D$ with $\mathcal{O}\overline{f} \cong f$.
For a space X, consider the family $\{\chi_x : \Omega X \to \mathbf{2}\}$ of characteristic functions.
A family $\{\overline{\chi_x} : \Omega_{\mathcal{C}} X \to \mathbf{2}_{\mathcal{C}}\}$ lifting it is *Cartesian* if it is universal among such lifts: whenever for $f : \mathcal{O}D \to \Omega X$ there are lifts for all $\chi_x \circ f$, then f lifts uniquely.

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For a space X, consider the family $\{\chi_x : \Omega X \to \mathbf{2}\}$ of characteristic functions.
A family $\{\overline{\chi_x} : \Omega_{\mathcal{C}} X \to \mathbf{2}_{\mathcal{C}}\}$ lifting it is *Cartesian* if it is universal among such lifts: whenever for $f : \mathcal{O}D \to \Omega X$ there are lifts for all $\chi_x \circ f$, then f lifts uniquely.

Cartesian-ness ensures that $\Omega_{\mathcal{C}} X$ has the analogue of the universal property of the spatialization of a frame.

SFC-categories: spatializable categories

Define $\mathbf{Top}_{\mathcal{C}} \subseteq \mathbf{Top}$ as the full subcategory of spaces admitting Cartesian lifts.
A PFC-category is *spatializable* (an *SFC-category*) if:

- $\mathbb{S} \in \mathbf{Top}_{\mathcal{C}}$;
- $\mathrm{pt}_{\mathcal{C}} C \in \mathbf{Top}_{\mathcal{C}}$ for all $C \in \mathcal{C}$.

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Following is the main theorem in Dimov and Tholen (1989) applied to our special case.

Theorem

Every SFC-category induces a natural dual adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=1.5ex, "\mathrm{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] & {\mathbf{Top}_{\mathcal{C}}}^{op} \arrow[l, shift left=1.5ex, "\Omega_{\mathcal{C}}"] \end{tikzcd}$$

via the dualizing object $(\mathbf{2}_{\mathcal{C}}, \mathbb{S})$.

Examples of SFC-categories

Example

All the FC-categories we have seen SFC-categories:

1. $\mathbf{Frm}_{\mathcal{E}} \subseteq \mathbf{Frm}$ yields T_D-duality via the construction described above, with $(\mathbf{2}, \mathbb{S})$ as the dualizing object;

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3. $\mathcal{O}_{\mathcal{M}} : \mathbf{MT} \to \mathbf{Frm}$ yields $\mathcal{P} : \mathbf{Top} \leftrightarrows \mathbf{MT} : \mathrm{pt}_{\mathcal{M}}$ via $(\mathbf{2}_{\mathcal{M}}, \mathbb{S})$;

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4. $\Omega : \mathbf{Top_0}^{\mathrm{op}} \to \mathbf{Frm}$ yields $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm} : \mathrm{pt}$ via $(\{*\}, \mathbb{S})$.

Thus all known dualities are instances of the same abstract framework.

Basic properties of SFC-categories

Proposition

For any SFC-category, the adjunction $\mathrm{pt}_{\mathcal{C}} \dashv \Omega_{\mathcal{C}}$ *is idempotent.*

Moreover, there is a right map of adjunctions:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "\mathrm{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] \arrow[d, "\mathcal{O}"'] & \mathbf{Top}_{\mathcal{C}}{}^{\mathrm{op}} \arrow[l, shift left, "\Omega_{\mathcal{C}}"] \arrow[d, hook] \\
\mathbf{Frm} \arrow[r, shift left, "\mathrm{pt}"] \arrow[r, phantom, "\perp"] & \mathbf{Top}^{\mathrm{op}} \arrow[l, shift left, "\Omega"]
\end{tikzcd}$$

The natural transformation $\xi : \mathrm{pt}_{\mathcal{C}} \implies \mathrm{pt}\mathcal{O}$ is a subspace inclusion.

Maps of SFC-categories

A *PFC-functor*

$$H : (\mathcal{O}_{\mathcal{C}}, \mathcal{C}, \mathbf{2}_{\mathcal{C}}) \to (\mathcal{O}_{\mathcal{D}}, \mathcal{D}, \mathbf{2}_{\mathcal{D}})$$

is a functor $H : \mathcal{C} \to \mathcal{D}$ satisfying $\mathcal{O}_{\mathcal{D}} H \cong \mathcal{O}_{\mathcal{C}}$ and $H(\mathbf{2}_{\mathcal{C}}) \cong \mathbf{2}_{\mathcal{D}}$.

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Proposition

Every SFC-functor induces a right map of adjunctions:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "\mathrm{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] \arrow[d, "H"'] & {\mathbf{Top}_{\mathcal{C}}}^{\mathrm{op}} \arrow[l, shift left, "\Omega_{\mathcal{C}}"] \arrow[d, hook] \\
\mathcal{D} \arrow[r, shift left, "\mathrm{pt}_{\mathcal{D}}"] \arrow[r, phantom, "\perp"] & \mathbf{Top}_{\mathcal{D}}^{\mathrm{op}} \arrow[l, shift left, "\Omega_{\mathcal{D}}"]
\end{tikzcd}$$

SFC-adjunctions

Definition

A *PFC-adjunction* is an adjunction

$$(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathbf{2}_{\mathcal{C}}) \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{D}, \mathcal{O}_{\mathcal{D}}, \mathbf{2}_{\mathcal{D}})$$

such that for the unit η and counit ϵ:

- $\mathcal{O}(\eta)$ and $\mathcal{O}(\epsilon)$ are frame isomorphisms;
- $\eta_{\mathbf{2}_{\mathcal{C}}}$ and $\epsilon_{\mathbf{2}_{\mathcal{D}}}$ are isomorphisms.

SFC-adjunctions

Definition

A *PFC-adjunction* is an adjunction

$$\begin{tikzcd}(\mathcal{C}, \mathcal{O}_\mathcal{C}, \mathbf{2}_\mathcal{C}) \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & (\mathcal{D}, \mathcal{O}_\mathcal{D}, \mathbf{2}_\mathcal{D}) \arrow[l, shift left=2, "U"]\end{tikzcd}$$

such that for the unit η and counit ϵ:

- $\mathcal{O}(\eta)$ and $\mathcal{O}(\epsilon)$ are frame isomorphisms;
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Proposition

Right PFC-adjoints preserve Cartesian lifts of spaces. In particular, they are SFC-functors.

Thus, adjunctions between PFC-categories give morphisms between induced dualities.

Fibers of an FC-category

For a frame L, the *fiber* $\mathcal{O}^{-1}(L)$ has:

- Objects: (C, θ^C) with θ^C and isomorphism $\mathcal{O}C \cong L$;
- Morphisms: $f : C \to D$ making the triangle commute:

$$\begin{tikzcd}
\mathcal{O}C \arrow[rr, "\mathcal{O}f"] \arrow[dr, "\theta^C(\cong)"'] & & \mathcal{O}D \arrow[dl, "\theta^D(\cong)"] \\
& L &
\end{tikzcd}$$

Since $\mathcal{O}$ is faithful, fibers are *preorders*.

Fiber-initial and fiber-terminal objects

Since fibers are preorders, we study:

- *Fiber-initial* objects (bottom elements);
- *Fiber-terminal* objects (top elements).

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Why do we care about these? In the classical case:

Theorem (Banaschewski and Pultr, 2010)

For the FC-category $\Omega : \mathbf{Top_0}^{\mathrm{op}} \to \mathbf{Frm}$*:*

- *Sober spaces are exactly the fiber-initial objects.*
- *T_D spaces are exactly the fiber-terminal objects.*

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Taking contravariance into account: a T_0 space X is sober iff it is the *largest* among spaces with isomorphic frame of opens; it is T_D iff it is the *smallest*.

Total SFC-categories

An SFC-category $(\mathcal{O}, \mathcal{C}, \mathbf{2}_{\mathcal{C}})$ is *total* if it satisfies:

1. $\mathrm{Fix}(\mathbf{Top}_{\mathcal{C}}) = \mathbf{Top_0}$: all T_0-spaces are captured as fixpoints;

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Axiom 4 is a version in our general setting of the following classical result.

Proposition

For a space X, every locally closed sublocale of the frame ΩX is induced by some (locally closed) subspace of X.

Total SFC-categories

Theorem

Raney *is a total SFC-category.*

Theorem

MT *satisfies 1, 2, and 4.*

The main theorem: fiber-initials

Let $(\mathcal{O}, \mathcal{C}, \mathbf{2}_\mathcal{C})$ be an SFC-category satisfying 1, 2, and 3. Define:

$$\mathcal{C}_\mathcal{I} := \text{ full subcategory of fiber-initial objects}$$

Theorem

There is an adjunction

$$\mathrm{pt}_\mathcal{C} : \mathcal{C}_\mathcal{I} \leftrightarrows \mathbf{Top} : \mathcal{F}\mathcal{O}\Omega_\mathcal{C}$$

whose fixpoints are exactly the sober spaces.

This shows that in any total SFC-category, the fiber-initial objects give a duality for sober spaces.

Example

In particular, this holds for **Raney**.

The main theorem: fiber-terminals

Let $(\mathcal{O}, \mathcal{C}, \mathbf{2}_{\mathcal{C}})$ be an SFC-category satisfying 1, 2, and 4. Define:

$$\mathcal{C}_{\mathcal{T}} := \text{ full subcategory of fiber-terminal objects}$$

Theorem

There is an adjunction

$$\mathrm{pt}_{\mathcal{C}} : \mathcal{C}_{\mathcal{T}} \leftrightarrows \mathbf{Top}_D : \Omega_{\mathcal{C}}$$

whose fixpoints are exactly the T_D spaces.

The fiber-terminal objects give a duality for T_D-spaces.

Example

In particular, this holds for **Raney** and **MT**.

The big picture

In a total SFC-category, we have a chain of adjunctions:

$$
\begin{tikzcd}
\mathcal{C}_{\mathcal{T}} \arrow[r, shift left=2, "\mathsf{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] \arrow[d, hook] & \mathbf{Top}_D \arrow[l, shift left=2, "\Omega_{\mathcal{C}}"] \arrow[d, hook] \\
\mathcal{C} \arrow[r, shift left=2, "\mathsf{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] \arrow[d, two heads, "\mathcal{O}\mathcal{F}"'] & \mathbf{Top} \arrow[l, shift left=2, "\Omega_{\mathcal{C}}"] \arrow[d, equal] \\
\mathcal{C}_{\mathcal{I}} \arrow[r, shift left=2, "\mathsf{pt}_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift left=2, "\mathcal{F}\mathcal{O}\Omega_{\mathcal{C}}"]
\end{tikzcd}
$$

$$\mathrm{Fix}(\mathbf{Top}_D) = \mathbf{Top}_D$$

$$\mathrm{Fix}(\mathbf{Top}) = \mathbf{Top_0}$$

$$\mathrm{Fix}(\mathbf{Top}) = \mathbf{Sob}$$

- Fiber-terminals give a *subobject*: $I_{\mathcal{T}} : \mathcal{C}_{\mathcal{T}} \hookrightarrow \mathcal{C}$ is an SFC-functor.
- Fiber-initials give a *quotient*: $\mathcal{F}\mathcal{O} : \mathcal{C} \to \mathcal{C}_{\mathcal{I}}$ is an SFC-functor.

Summary

We have introduced:

- **SFC-categories**: A unified framework for pointfree T_0 spaces.
- **Their dual adjunctions**: Every SFC-category induces $\mathrm{pt}_{\mathcal{C}} \dashv \Omega_{\mathcal{C}}$.
- **Morphisms**: SFC-functors induce morphisms between adjunctions.
- **Total SFC-categories**: Satisfying four natural axioms that ensure that:
  - Fiber-initial objects give sober spaces (classical duality);
  - Fiber-terminal objects give T_D spaces (Banaschewski–Pultr).

Key examples

Raney extensions, MT-algebras.

Thus we have a single framework that unifies and generalizes the known pointfree dualities for T_0-spaces.

Outlook

- **Possible applications:**
  - Domain theory: pointfree treatment of the Scott topology (e.g. frames whose spectrum is a Scott space are difficult to characterize; what about objects of a total SFC-categories?);
  - Pointfree (stable) compactifications of T_0 spaces.

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  - Domain theory: pointfree treatment of the Scott topology (e.g. frames whose spectrum is a Scott space are difficult to characterize; what about objects of a total SFC-categories?);
  - Pointfree (stable) compactifications of T_0 spaces.
- **Open questions:**
  - What other dualities fit into (some variation of) this framework?
  - Is there a "universal" total SFC category? This would give us a canonical pointfree representation of T_0-spaces.

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