Every join semilattice with 0 is isomorphic to the semilattice of compact open sets of some space

Marcus Tressl, Manchester, UK

TACL 2026
July 21-31, 2026, Kraków

Many thanks to the program committee for putting together two wonderful weeks and to the organizers of the school and the conference:

Nick Bezhanishvili

Nick Galatos

Zalán Gyenis

Tomasz Kowalski

Katarzyna Słomczyńska

In this talk, **semilattice** always means 'join semilattice' and **space** always means 'topological space satisfying the T_0-axiom'.

1. Introduction - Statement of the Theorem

In this talk, **semilattice** always means 'join semilattice' and **space** always means 'topological space satisfying the T_0-axiom'.

Definition For a space X we write

$$\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\},$$

where 'compact open' means 'compact <u>and</u> open'.

1. Introduction - Statement of the Theorem

In this talk, **semilattice** always means 'join semilattice' and **space** always means 'topological space satisfying the T_0-axiom'.

Definition For a space X we write

$$\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\},$$

where 'compact open' means 'compact <u>and</u> open'.

Obviously $\mathring{\mathcal{K}}(X)$ is a semilattice with smallest element $\emptyset$, where joins are unions.

1. Introduction - Statement of the Theorem

In this talk, **semilattice** always means 'join semilattice' and **space** always means 'topological space satisfying the T_0-axiom'.

Definition For a space X we write

$$\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\},$$

where 'compact open' means 'compact and open'.

Obviously $\mathring{\mathcal{K}}(X)$ is a semilattice with smallest element $\emptyset$, where joins are unions.

Theorem Every semilattice with 0 is of this form (in a functorial way).

1. Introduction - Statement of the Theorem

In this talk, **semilattice** always means 'join semilattice' and **space** always means 'topological space satisfying the T_0-axiom'.

Definition For a space X we write

$$\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\},$$

where 'compact open' means 'compact <u>and</u> open'.

Obviously $\mathring{\mathcal{K}}(X)$ is a semilattice with smallest element $\emptyset$, where joins are unions.

Theorem Every semilattice with 0 is of this form (in a functorial way).

This talk is about the construction of such a space, after an explanation why I am interested in the statement.

1. Introduction - History, seen from topology

Let X be a space. $\overset{\circ}{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\}$,

1. Introduction - History, seen from topology

Let X be a space. $\overset{\circ}{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\}$,

1. If X is Boolean ($=$compact T_2 and totally disconnected), then $\overset{\circ}{\mathcal{K}}(X)$ is the Boolean algebra of clopen subsets of X and all Boolean algebras appear (M. Stone).

1. Introduction - History, seen from topology

Let X be a space. $\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\}$,

1. If X is Boolean (=compact T_2 and totally disconnected), then $\mathring{\mathcal{K}}(X)$ is the Boolean algebra of clopen subsets of X and all Boolean algebras appear (M. Stone).
2. If X is spectral [Hoc69; DST19], then $\mathring{\mathcal{K}}(X)$ is a bounded distributive lattice and all bounded distributive lattices appear (M. Stone).

1. Introduction - History, seen from topology

Let X be a space. $\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\}$,

1. If X is Boolean (=compact T_2 and totally disconnected), then $\mathring{\mathcal{K}}(X)$ is the Boolean algebra of clopen subsets of X and all Boolean algebras appear (M. Stone).
2. If X is spectral [Hoc69; DST19], then $\mathring{\mathcal{K}}(X)$ is a bounded distributive lattice and all bounded distributive lattices appear (M. Stone).
3. If X is **compactly based**, i.e. $\mathring{\mathcal{K}}(X)$ is a basis, then $\mathring{\mathcal{K}}(X)$ is a **distributive semilattice** and every distributive semilattice with 0 is of this form (G. Grätzer, [Grä98, II.5]).

1. Introduction - History, seen from topology

Let X be a space. $\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\}$,

1. If X is Boolean (=compact T_2 and totally disconnected), then $\mathring{\mathcal{K}}(X)$ is the Boolean algebra of clopen subsets of X and all Boolean algebras appear (M. Stone).
2. If X is spectral [Hoc69; DST19], then $\mathring{\mathcal{K}}(X)$ is a bounded distributive lattice and all bounded distributive lattices appear (M. Stone).
3. If X is **compactly based**, i.e. $\mathring{\mathcal{K}}(X)$ is a basis, then $\mathring{\mathcal{K}}(X)$ is a **distributive semilattice** and every distributive semilattice with 0 is of this form (G. Grätzer, [Grä98, II.5]).

   Recall that a semilattice is called distributive if it satisfies
   $$\forall\, c \leq a \vee b\ \exists\, a' \leq a, b' \leq b : c = a' \vee b'.$$

1. Introduction - History, seen from topology

Let X be a space. $\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\}$,

1. If X is Boolean (=compact T_2 and totally disconnected), then $\mathring{\mathcal{K}}(X)$ is the Boolean algebra of clopen subsets of X and all Boolean algebras appear (M. Stone).
2. If X is spectral [Hoc69; DST19], then $\mathring{\mathcal{K}}(X)$ is a bounded distributive lattice and all bounded distributive lattices appear (M. Stone).
3. If X is **compactly based**, i.e. $\mathring{\mathcal{K}}(X)$ is a basis, then $\mathring{\mathcal{K}}(X)$ is a **distributive semilattice** and every distributive semilattice with 0 is of this form (G. Grätzer, [Grä98, II.5]).

   Recall that a semilattice is called distributive if it satisfies
   $$\forall\, c \leq a \vee b \;\exists\, a' \leq a, b' \leq b : c = a' \vee b'.$$

   Phrased topologically:
   $$X \text{ is compactly based} \iff \mathcal{O}(X) \text{ is an algebraic frame.}$$

1. Introduction - History, seen from topology

Let X be a space. $\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ compact open}\}$,

1. If X is Boolean (=compact T_2 and totally disconnected), then $\mathring{\mathcal{K}}(X)$ is the Boolean algebra of clopen subsets of X and all Boolean algebras appear (M. Stone).
2. If X is spectral [Hoc69; DST19], then $\mathring{\mathcal{K}}(X)$ is a bounded distributive lattice and all bounded distributive lattices appear (M. Stone).
3. If X is **compactly based**, i.e. $\mathring{\mathcal{K}}(X)$ is a basis, then $\mathring{\mathcal{K}}(X)$ is a **distributive semilattice** and every distributive semilattice with 0 is of this form (G. Grätzer, [Grä98, II.5]).

   Recall that a semilattice is called distributive if it satisfies
   $$\forall\, c \leq a \vee b \;\exists\, a' \leq a, b' \leq b : c = a' \vee b'.$$

   Phrased topologically:
   $$X \text{ is compactly based} \iff \mathcal{O}(X) \text{ is an algebraic frame.}$$

There are several papers extending the dualities above beyond the distributive case, going a different route to what we are after. For example see (the fully incomplete list) [BCM25; CG20; MJ14].

Algebraic Logic. The Tarski-Lindenbaum algebra of a given (propositional) logic is a semilattice and a representation theorem as above establishes a completeness theorem of the logic: The points of X are seen as truth assignments.

1. Introduction - my interest in the Theorem

Algebraic Logic. The Tarski-Lindenbaum algebra of a given (propositional) logic is a semilattice and a representation theorem as above establishes a completeness theorem of the logic: The points of X are seen as truth assignments.

Model Theory. First order structures sometimes come in the form of global sections of sheaves of simpler structures over some base space X. Prominent example: Commutative rings or abelian ℓ-groups are global sections of their 'Zariski-like' sheaf. 'Simpler structures' here refers to fields or totally ordered abelian groups.

1. Introduction - my interest in the Theorem

Algebraic Logic. The Tarski-Lindenbaum algebra of a given (propositional) logic is a semilattice and a representation theorem as above establishes a completeness theorem of the logic: The points of X are seen as truth assignments.

Model Theory. First order structures sometimes come in the form of global sections of sheaves of simpler structures over some base space X. Prominent example: Commutative rings or abelian ℓ-groups are global sections of their 'Zariski-like' sheaf. 'Simpler structures' here refers to fields or totally ordered abelian groups.

To analyze such a structure, a naïve approach is:
Study (a) the stalks, (b) the base space and then (c) glue the information.

1. Introduction - my interest in the Theorem

Algebraic Logic. The Tarski-Lindenbaum algebra of a given (propositional) logic is a semilattice and a representation theorem as above establishes a completeness theorem of the logic: The points of X are seen as truth assignments.

Model Theory. First order structures sometimes come in the form of global sections of sheaves of simpler structures over some base space X. Prominent example: Commutative rings or abelian ℓ-groups are global sections of their 'Zariski-like' sheaf. 'Simpler structures' here refers to fields or totally ordered abelian groups.

To analyze such a structure, a naïve approach is:
Study (a) the stalks, (b) the base space and then (c) glue the information.

How to study a space X as a first order structure? The lattice of **all** open sets is mostly way too complicated, however $\mathring{\mathcal{K}}(X)$ is often accessible.

1. Introduction - my interest in the Theorem

Geometric Algebra Let K be a field and let $J = (J, \subseteq)$ be the poset of K-subvector spaces of K^3.

1. Introduction - my interest in the Theorem

Geometric Algebra   Let K be a field and let $J = (J, \subseteq)$ be the poset of K-subvector spaces of K^3.

Geometric Algebra implies that the partially ordered set $(J, \subseteq)$ is bi-interpretable with the field $(K, +, \times)$. [Hod93, 5.4(c), Example 1]

1. Introduction - my interest in the Theorem

Geometric Algebra Let K be a field and let $J = (J, \subseteq)$ be the poset of K-subvector spaces of K^3.

Geometric Algebra implies that the partially ordered set $(J, \subseteq)$ is bi-interpretable with the field $(K, +, \times)$. [Hod93, 5.4(c), Example 1]

Now J is a complete and modular lattice, but it is not distributive, so none of the examples in the history above can be used to represent J as $\mathring{\mathcal{K}}(X)$.

1. Introduction - my interest in the Theorem

Geometric Algebra   Let K be a field and let $J = (J, \subseteq)$ be the poset of K-subvector spaces of K^3.

Geometric Algebra implies that the partially ordered set $(J, \subseteq)$ is bi-interpretable with the field $(K, +, \times)$. [Hod93, 5.4(c), Example 1]

Now J is a complete and modular lattice, but it is not distributive, so none of the examples in the history above can be used to represent J as $\overset{\circ}{\mathcal{K}}(X)$.

Another example of a semilattice is the set J of principal open subsets of the prime spectrum of a ring. One wishes to have a version of the prime spectrum whose compact opens are the <u>principal</u> open sets.

1. Introduction - my interest in the Theorem

Geometric Algebra Let K be a field and let $J = (J, \subseteq)$ be the poset of K-subvector spaces of K^3.

Geometric Algebra implies that the partially ordered set $(J, \subseteq)$ is bi-interpretable with the field $(K, +, \times)$. [Hod93, 5.4(c), Example 1]

Now J is a complete and modular lattice, but it is not distributive, so none of the examples in the history above can be used to represent J as $\mathring{\mathcal{K}}(X)$.

Another example of a semilattice is the set J of principal open subsets of the prime spectrum of a ring. One wishes to have a version of the prime spectrum whose compact opens are the <u>principal</u> open sets.

Yet another example of a semilattice is the set J of basic semi-algebraic subsets of $\mathbb{R}^n$. One wishes to have a version of the real spectrum whose compact opens are the <u>basic</u> open semi-algebraic sets [BCR98, Def. 2.7.1].

.

2. Core idea of the construction

Recall: Given a semilattice J with 0 we want a space X with $J \cong \mathring{\mathcal{K}}(X)$.

2. Core idea of the construction

Recall: Given a semilattice J with 0 we want a space X with $J \cong \overset{\circ}{\mathcal{K}}(X)$.

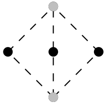
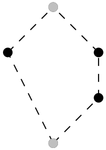
The matter is already a problem for finite J. If J is not distributive, we need to construct an infinite space, because for a finite space X, the poset $\overset{\circ}{\mathcal{K}}(X)$ is the complete and distributive lattice of all open subsets of X.

2. Core idea of the construction

Recall: Given a semilattice J with 0 we want a space X with $J \cong \mathring{\mathcal{K}}(X)$.

The matter is already a problem for finite J. If J is not distributive, we need to construct an infinite space, because for a finite space X, the poset $\mathring{\mathcal{K}}(X)$ is the complete and distributive lattice of all open subsets of X.

Recall that a lattice is distributive just if it does neither contain the lattice

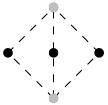
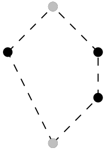
M_3:  , nor the lattice N_5:  as sublattices.

2. Core idea of the construction

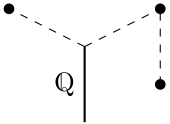
Recall: Given a semilattice J with 0 we want a space X with $J \cong \mathring{\mathcal{K}}(X)$.

The matter is already a problem for finite J. If J is not distributive, we need to construct an infinite space, because for a finite space X, the poset $\mathring{\mathcal{K}}(X)$ is the complete and distributive lattice of all open subsets of X.

Recall that a lattice is distributive just if it does neither contain the lattice

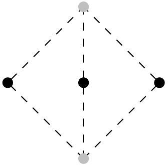
M_3:  , nor the lattice N_5:  as sublattices.

Solution for N_5: The lattice N_5 is isomorphic to $\mathring{\mathcal{K}}(P, \tau^\ell)$, where τ^ℓ is the (coarse) **lower topology** of the following poset P:

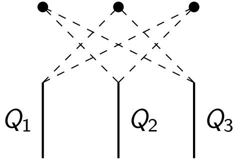
$P \;=$ 

A subbasis of τ^ℓ is the set
$\{P \setminus a^\uparrow \mid a \in P\}$

Here is how one can get a space X with $\mathrm{M}_3 \cong \mathring{\mathcal{K}}(X)$.



M_3



A poset P with $\mathrm{M}_3 \cong \mathring{\mathcal{K}}(P, \tau^\ell)$

The Q_i here are disjoint copies of the chain $\mathbb{Q}$, or in fact any dense linear order without endpoints.

3. Auxiliary Constructions - The poset Aux(P)

From now on we fix a dense totally ordered set I without endpoints.

3. Auxiliary Constructions - The poset Aux(P)

From now on we fix a dense totally ordered set I without endpoints.
Let P be a poset. We write $\mathrm{ac}(P)$ for the set of finite anti-chains of P of size ≥ 2. We write $Q = I \times \mathrm{ac}(P)$ and $Q_s := I \times \{s\}$. Think of Q as the disjoint union of copies Q_s of the poset I.

3. Auxiliary Constructions - The poset Aux(P)

From now on we fix a dense totally ordered set I without endpoints.
Let P be a poset. We write $\mathrm{ac}(P)$ for the set of finite anti-chains of P of size ≥ 2. We write $Q = I \times \mathrm{ac}(P)$ and $Q_s := I \times \{s\}$. Think of Q as the disjoint union of copies Q_s of the poset I.

We define a poset $(\mathrm{Aux}(P), \leq)$ as follows. The underlying set is $\mathrm{Aux}(P) = P \dot{\cup} Q$. The strict partial order $<$ of $\mathrm{Aux}(P)$ is defined as follows (where $s \in \mathrm{ac}(P)$ and up-sets and lower bounds are to be read in P).

3. Auxiliary Constructions - The poset Aux(P)

From now on we fix a dense totally ordered set I without endpoints.
Let P be a poset. We write $\mathrm{ac}(P)$ for the set of finite anti-chains of P of size ≥ 2. We write $Q = I \times \mathrm{ac}(P)$ and $Q_s := I \times \{s\}$. Think of Q as the disjoint union of copies Q_s of the poset I.

We define a poset $(\mathrm{Aux}(P), \leq)$ as follows. The underlying set is $\mathrm{Aux}(P) = P \dot\cup Q$. The strict partial order $<$ of $\mathrm{Aux}(P)$ is defined as follows (where $s \in \mathrm{ac}(P)$ and up-sets and lower bounds are to be read in P).

(A) On P and on each Q_s, the relation $<$ is the given strict partial order.

3. Auxiliary Constructions - The poset Aux(P)

From now on we fix a dense totally ordered set I without endpoints.
Let P be a poset. We write $\mathrm{ac}(P)$ for the set of finite anti-chains of P of size ≥ 2. We write $Q = I \times \mathrm{ac}(P)$ and $Q_s := I \times \{s\}$. Think of Q as the disjoint union of copies Q_s of the poset I.

We define a poset $(\mathrm{Aux}(P), \leq)$ as follows. The underlying set is $\mathrm{Aux}(P) = P \dot\cup Q$. The strict partial order $<$ of $\mathrm{Aux}(P)$ is defined as follows (where $s \in \mathrm{ac}(P)$ and up-sets and lower bounds are to be read in P).

(A) On P and on each Q_s, the relation $<$ is the given strict partial order.

(B) If $q \in Q_s$ and $p \in P$, then
$(\mathrm{B})_1 \quad p < q \iff p \in \mathrm{Lb}(s)$.

3. Auxiliary Constructions - The poset Aux(P)

From now on we fix a dense totally ordered set I without endpoints.
Let P be a poset. We write $\mathrm{ac}(P)$ for the set of finite anti-chains of P of size ≥ 2. We write $Q = I \times \mathrm{ac}(P)$ and $Q_s := I \times \{s\}$. Think of Q as the disjoint union of copies Q_s of the poset I.

We define a poset $(\mathrm{Aux}(P), \leq)$ as follows. The underlying set is $\mathrm{Aux}(P) = P \dot\cup Q$. The strict partial order $<$ of $\mathrm{Aux}(P)$ is defined as follows (where $s \in \mathrm{ac}(P)$ and up-sets and lower bounds are to be read in P).

(A) On P and on each Q_s, the relation $<$ is the given strict partial order.

(B) If $q \in Q_s$ and $p \in P$, then
  $(B)_1 \quad p < q \iff p \in \mathrm{Lb}(s)$.
  $(B)_2 \quad q < p \iff p \in s^{\uparrow}$.

3. Auxiliary Constructions - The poset Aux(P)

From now on we fix a dense totally ordered set I without endpoints.
Let P be a poset. We write $\mathrm{ac}(P)$ for the set of finite anti-chains of P of size ≥ 2. We write $Q = I \times \mathrm{ac}(P)$ and $Q_s := I \times \{s\}$. Think of Q as the disjoint union of copies Q_s of the poset I.

We define a poset $(\mathrm{Aux}(P), \leq)$ as follows. The underlying set is $\mathrm{Aux}(P) = P \dot{\cup} Q$. The strict partial order $<$ of $\mathrm{Aux}(P)$ is defined as follows (where $s \in \mathrm{ac}(P)$ and up-sets and lower bounds are to be read in P).

(A) On P and on each Q_s, the relation $<$ is the given strict partial order.

(B) If $q \in Q_s$ and $p \in P$, then
   $(B)_1$ $\quad p < q \iff p \in \mathrm{Lb}(s)$.
   $(B)_2$ $\quad q < p \iff p \in s^{\uparrow}$.

(C) For $q \in Q_s, q' \in Q_{s'}$ with $s \neq s'$ we take
$$q < q' \quad \iff s^{\uparrow} \cap \mathrm{Lb}(s') \neq \emptyset.$$

3. Auxiliary Constructions - The poset Aux(P)

From now on we fix a dense totally ordered set I without endpoints.
Let P be a poset. We write $\mathrm{ac}(P)$ for the set of finite anti-chains of P of size ≥ 2. We write $Q = I \times \mathrm{ac}(P)$ and $Q_s := I \times \{s\}$. Think of Q as the disjoint union of copies Q_s of the poset I.

We define a poset $(\mathrm{Aux}(P), \leq)$ as follows. The underlying set is $\mathrm{Aux}(P) = P \dot\cup Q$. The strict partial order $<$ of $\mathrm{Aux}(P)$ is defined as follows (where $s \in \mathrm{ac}(P)$ and up-sets and lower bounds are to be read in P).

(A) On P and on each Q_s, the relation $<$ is the given strict partial order.

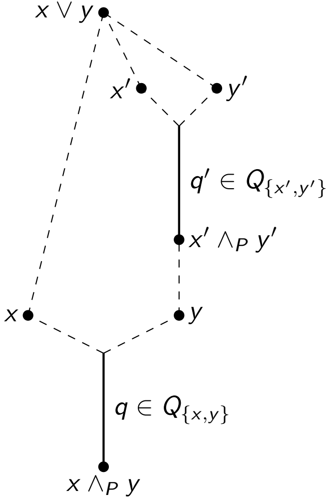
(B) If $q \in Q_s$ and $p \in P$, then
- $(B)_1$ $\quad p < q \iff p \in \mathrm{Lb}(s)$.
- $(B)_2$ $\quad q < p \iff p \in s^{\uparrow}$.

(C) For $q \in Q_s, q' \in Q_{s'}$ with $s \neq s'$ we take
$$q < q' \quad \iff s^{\uparrow} \cap \mathrm{Lb}(s') \neq \emptyset.$$

One verifies without difficulty that $<$ is indeed a strict partial order and the resulting poset $\mathrm{Aux}(P)$ has P and all Q_s as subposets.

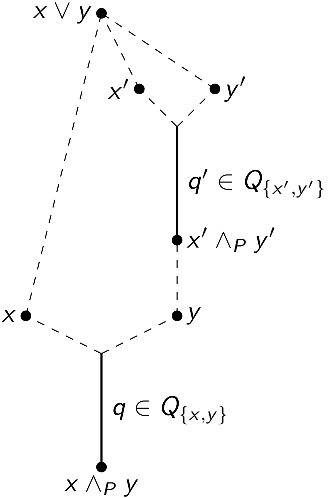
3. Auxiliary Constructions - The poset Aux(P)

Here is a picture of the core of this definition for $\{x, y\}, \{x', y'\} \in \mathrm{ac}(P)$, assuming P is a lattice and $y \leq x', y' \leq x \vee y$. :



3. Auxiliary Constructions - The poset Aux(P)

Here is a picture of the core of this definition for $\{x, y\}, \{x', y'\} \in \mathrm{ac}(P)$, assuming P is a lattice and $y \leq x', y' \leq x \vee y$. :



No nontrivial finite meet exists anymore!

3. Auxiliary Constructions - The coarse lower topology

Definition For a poset X, let $\tau^\ell(X)$ be the (coarse) **lower topology** of X which has the sets $X \setminus x^\uparrow$ as a subbasis of open sets. Hence every open set is a down-set and $\tau^\ell(X)$ is the coarsest T_0-topology on X with this property.

3. Auxiliary Constructions - The coarse lower topology

Definition For a poset X, let $\tau^\ell(X)$ be the (coarse) **lower topology** of X which has the sets $X \setminus x^\uparrow$ as a subbasis of open sets. Hence every open set is a down-set and $\tau^\ell(X)$ is the coarsest T_0-topology on X with this property.

This topology is 'intrinsic' in the sense that it is defined in terms of the order relation only. It looks a priori a bit artificial to consider this topology, however it occurs everywhere in mathematics because of the following

Definition For a poset X, let $\tau^\ell(X)$ be the (coarse) **lower topology** of X which has the sets $X \setminus x^\uparrow$ as a subbasis of open sets. Hence every open set is a down-set and $\tau^\ell(X)$ is the coarsest T_0-topology on X with this property.

This topology is 'intrinsic' in the sense that it is defined in terms of the order relation only. It looks a priori a bit artificial to consider this topology, however it occurs everywhere in mathematics because of the following

Observation If S is any space, then the natural map

$$\begin{aligned} S &\longrightarrow \mathcal{O}(S) \\ s &\longmapsto S \setminus s^\uparrow \end{aligned}$$

is an embedding when the topology $\mathcal{O}(S)$ itself is endowed with the coarse lower topology defined by the partial order $\subseteq$.

3. Auxiliary Constructions - The space Aux(P)

Definition For a poset P we define the **space** $\mathrm{Aux}(P)$ as the set $\mathrm{Aux}(P)$ endowed with the coarse lower topology of the poset $\mathrm{Aux}(P)$.

3. Auxiliary Constructions - The space Aux(P)

Definition For a poset P we define the **space** $\mathrm{Aux}(P)$ as the set $\mathrm{Aux}(P)$ endowed with the coarse lower topology of the poset $\mathrm{Aux}(P)$.

Proposition
Let P be a conditional join semilattice and write $X = \mathrm{Aux}(P)$.

3. Auxiliary Constructions - The space Aux(P)

Definition   For a poset P we define the **space** $\mathrm{Aux}(P)$ as the set $\mathrm{Aux}(P)$ endowed with the coarse lower topology of the poset $\mathrm{Aux}(P)$.

Proposition
Let P be a conditional join semilattice and write $X = \mathrm{Aux}(P)$. Then

(i) $\mathring{\mathcal{K}}(X) \subseteq \{\emptyset, X\} \cup \{X \setminus p^{\uparrow} \mid p \in P\}$.

3. Auxiliary Constructions - The space Aux(P)

Definition   For a poset P we define the **space** $\mathrm{Aux}(P)$ as the set $\mathrm{Aux}(P)$ endowed with the coarse lower topology of the poset $\mathrm{Aux}(P)$.

Proposition
Let P be a conditional join semilattice and write $X = \mathrm{Aux}(P)$. Then

(i) $\mathring{\mathcal{K}}(X) \subseteq \{\emptyset, X\} \cup \{X \setminus p^\uparrow \mid p \in P\}$.

(ii) For $p \in P$ the set $X \setminus p^\uparrow$ is compact for $\tau^\ell(X)$ if and only if $P \setminus p^\uparrow$ is compact for $\tau^\ell(P)$. In particular

$$\{X \setminus p^\uparrow \mid p \in P \text{ and } P \setminus p^\uparrow \in \mathring{\mathcal{K}}(P)\} \subseteq \mathring{\mathcal{K}}(X).$$

Here all up-sets are taken in X.

3. Auxiliary Constructions - The space Aux(P)

Definition For a poset P we define the **space** $\mathrm{Aux}(P)$ as the set $\mathrm{Aux}(P)$ endowed with the coarse lower topology of the poset $\mathrm{Aux}(P)$.

Proposition
Let P be a conditional join semilattice and write $X = \mathrm{Aux}(P)$. Then

(i) $\mathring{\mathcal{K}}(X) \subseteq \{\emptyset, X\} \cup \{X \setminus p^{\uparrow} \mid p \in P\}$.

(ii) For $p \in P$ the set $X \setminus p^{\uparrow}$ is compact for $\tau^{\ell}(X)$ if and only if $P \setminus p^{\uparrow}$ is compact for $\tau^{\ell}(P)$. In particular

$$\{X \setminus p^{\uparrow} \mid p \in P \text{ and } P \setminus p^{\uparrow} \in \mathring{\mathcal{K}}(P)\} \subseteq \mathring{\mathcal{K}}(X).$$

Here all up-sets are taken in X.

¡ This almost finishes the task when P is finite !

4. First construction of a space

Let J be a semilattice with 0 and define $L = \mathrm{Ideals}(J)$. Recall that L is an algebraic lattice and it is an algebraic frame if and only if J is a distributive semilattice.

We have a semilattice embedding $i : J \longrightarrow L,\ i(a) = a^{\downarrow}$ onto the sub-semilattice $K(L)$ of compact elements of L.

4. First construction of a space

Let J be a semilattice with 0 and define $L = \mathrm{Ideals}(J)$. Recall that L is an algebraic lattice and it is an algebraic frame if and only if J is a distributive semilattice.

We have a semilattice embedding $i : J \longrightarrow L,\ i(a) = a^{\downarrow}$ onto the sub-semilattice $K(L)$ of compact elements of L.

We define $P := L \setminus \{1_L\}$ and a space $X = \mathrm{Aux}(P)$.
Recall that X is the set $\mathrm{Aux}(P) = P \,\dot\cup\, (I \times \mathrm{ac}(P))$ and X carries the coarse lower topology of the poset $\mathrm{Aux}(P)$.

4. First construction of a space

Let J be a semilattice with 0 and define $L = \mathrm{Ideals}(J)$. Recall that L is an algebraic lattice and it is an algebraic frame if and only if J is a distributive semilattice.

We have a semilattice embedding $i : J \longrightarrow L,\ i(a) = a^{\downarrow}$ onto the sub-semilattice $K(L)$ of compact elements of L.

We define $P := L \setminus \{1_L\}$ and a space $X = \mathrm{Aux}(P)$.
Recall that X is the set $\mathrm{Aux}(P) = P \cup (I \times \mathrm{ac}(P))$ and X carries the coarse lower topology of the poset $\mathrm{Aux}(P)$.

Then the map

$$\begin{aligned} J &\longrightarrow \mathring{\mathcal{K}}(X) \\ a &\longmapsto X \setminus i(a)^{\uparrow} \end{aligned}$$

is a poset isomorphism – **which settles the theorem of the title**.

4. First construction of a space

Let J be a semilattice with 0 and define $L = \mathrm{Ideals}(J)$. Recall that L is an algebraic lattice and it is an algebraic frame if and only if J is a distributive semilattice.

We have a semilattice embedding $i : J \longrightarrow L,\ i(a) = a^{\downarrow}$ onto the sub-semilattice $K(L)$ of compact elements of L.

We define $P := L \setminus \{1_L\}$ and a space $X = \mathrm{Aux}(P)$.
Recall that X is the set $\mathrm{Aux}(P) = P \cup (I \times \mathrm{ac}(P))$ and X carries the coarse lower topology of the poset $\mathrm{Aux}(P)$.

Then the map

$$\begin{aligned} J &\longrightarrow \mathring{\mathcal{K}}(X) \\ a &\longmapsto X \setminus i(a)^{\uparrow} \end{aligned}$$

is a poset isomorphism – **which settles the theorem of the title**.

Proof: Apply the proposition and use the observation that an element c of L is a compact element of L iff the set $L \setminus c^{\uparrow}$ is compact for $\tau^{\ell}(L)$. $\square$

5. An extended (and better) construction

The construction above returns an infinite space when given a finite lattice J that is not a chain. We don't want that: It should in general return the old Stone-Grätzer dual, when J is a distributive semilattice. We remedy this now.

5. An extended (and better) construction

The construction above returns an infinite space when given a finite lattice J that is not a chain. We don't want that: It should in general return the old Stone-Grätzer dual, when J is a distributive semilattice. We remedy this now.

Let J be a semilattice with 0. As explained above, we may assume that $J = K(L)$, the subsemilattice of compact elements of a given algebraic lattice (so that we don't need to think about the embedding $i : J \longrightarrow L$).

5. An extended (and better) construction

Main Theorem Let L be an algebraic lattice and set $P := L \backslash \{\top\}$. Choose a set $Y \subseteq L$. (Think of Y as the set of irreducible elements). We define

$$S := \{s \in \mathrm{ac}(P) \mid s \subseteq K(L) \text{ and } \exists\, p \in Y : \bigwedge\nolimits_L s \leq p \notin s^{\uparrow}\}.$$

[1]Observe that this is in general very different from the coarse lower topology of Ξ: Think of the case when L is distributive and $Y := \mathrm{Irr}(L)$; then $S = \emptyset$ and $\Xi := \mathrm{Irr}(L)$ is the Stone-Grätzer dual of $K(L)$.

5. An extended (and better) construction

Main Theorem Let L be an algebraic lattice and set $P := L \setminus \{\top\}$. Choose a set $Y \subseteq L$. (Think of Y as the set of irreducible elements). We define

$$S := \{s \in \mathrm{ac}(P) \mid s \subseteq K(L) \text{ and } \exists\, p \in Y : \bigwedge\nolimits_L s \leq p \notin s^{\uparrow}\}.$$

Let $\Xi \subseteq \mathrm{Aux}(P)$ be such that the following conditions hold.

(a) For all $s \in S$ we have $Q_s \subseteq \Xi$.

---

[1] Observe that this is in general very different from the coarse lower topology of Ξ: Think of the case when L is distributive and $Y := \mathrm{Irr}(L)$; then $S = \emptyset$ and $\Xi := \mathrm{Irr}(L)$ is the Stone-Grätzer dual of $K(L)$.

5. An extended (and better) construction

Main Theorem Let L be an algebraic lattice and set $P := L \backslash \{\top\}$. Choose a set $Y \subseteq L$. (Think of Y as the set of irreducible elements). We define

$$S := \{s \in \mathrm{ac}(P) \mid s \subseteq K(L) \text{ and } \exists\, p \in Y : \bigwedge\nolimits_L s \leq p \notin s^{\uparrow}\}.$$

Let $\Xi \subseteq \mathrm{Aux}(P)$ be such that the following conditions hold.

(a) For all $s \in S$ we have $Q_s \subseteq \Xi$.

(b) For all $s \in \mathrm{ac}(P)$ we have $Q_s \subseteq \Xi$ or $Q_s \cap \Xi = \emptyset$.

[1]Observe that this is in general very different from the coarse lower topology of Ξ: Think of the case when L is distributive and $Y := \mathrm{Irr}(L)$; then $S = \emptyset$ and $\Xi := \mathrm{Irr}(L)$ is the Stone-Grätzer dual of $K(L)$.

5. An extended (and better) construction

Main Theorem Let L be an algebraic lattice and set $P := L \setminus \{\top\}$. Choose a set $Y \subseteq L$. (Think of Y as the set of irreducible elements). We define

$$S := \{s \in \mathrm{ac}(P) \mid s \subseteq K(L) \text{ and } \exists\, p \in Y : \bigwedge\nolimits_L s \leq p \notin s^{\uparrow}\}.$$

Let $\Xi \subseteq \mathrm{Aux}(P)$ be such that the following conditions hold.

(a) For all $s \in S$ we have $Q_s \subseteq \Xi$.

(b) For all $s \in \mathrm{ac}(P)$ we have $Q_s \subseteq \Xi$ or $Q_s \cap \Xi = \emptyset$.

(c) $\{p \in P \mid p \text{ completely irreducible}\} \subseteq \Xi \cap P \subseteq Y$.

[1] Observe that this is in general very different from the coarse lower topology of Ξ: Think of the case when L is distributive and $Y := \mathrm{Irr}(L)$; then $S = \emptyset$ and $\Xi := \mathrm{Irr}(L)$ is the Stone-Grätzer dual of $K(L)$.

5. An extended (and better) construction

Main Theorem Let L be an algebraic lattice and set $P := L \backslash \{\top\}$. Choose a set $Y \subseteq L$. (Think of Y as the set of irreducible elements). We define

$$S := \{s \in \mathrm{ac}(P) \mid s \subseteq K(L) \text{ and } \exists\, p \in Y : \bigwedge\nolimits_L s \leq p \notin s^{\uparrow}\}.$$

Let $\Xi \subseteq \mathrm{Aux}(P)$ be such that the following conditions hold.

(a) For all $s \in S$ we have $Q_s \subseteq \Xi$.

(b) For all $s \in \mathrm{ac}(P)$ we have $Q_s \subseteq \Xi$ or $Q_s \cap \Xi = \emptyset$.

(c) $\{p \in P \mid p \text{ completely irreducible}\} \subseteq \Xi \cap P \subseteq Y$.

We furnish Ξ with the subspace topology inherited from $(\mathrm{Aux}(P), \tau^{\ell})$.[1]

[1] Observe that this is in general very different from the coarse lower topology of Ξ: Think of the case when L is distributive and $Y := \mathrm{Irr}(L)$; then $S = \emptyset$ and $\Xi := \mathrm{Irr}(L)$ is the Stone-Grätzer dual of $K(L)$.

5. An extended (and better) construction

Main Theorem Let L be an algebraic lattice and set $P := L\backslash\{\top\}$. Choose a set $Y \subseteq L$. (Think of Y as the set of irreducible elements). We define

$$S := \{s \in \mathrm{ac}(P) \mid s \subseteq K(L) \text{ and } \exists\, p \in Y : \bigwedge\nolimits_L s \leq p \notin s^{\uparrow}\}.$$

Let $\Xi \subseteq \mathrm{Aux}(P)$ be such that the following conditions hold.

(a) For all $s \in S$ we have $Q_s \subseteq \Xi$.

(b) For all $s \in \mathrm{ac}(P)$ we have $Q_s \subseteq \Xi$ or $Q_s \cap \Xi = \emptyset$.

(c) $\{p \in P \mid p \text{ completely irreducible}\} \subseteq \Xi \cap P \subseteq Y$.

We furnish Ξ with the subspace topology inherited from $(\mathrm{Aux}(P), \tau^{\ell})$.[1]

Then the map $\gamma : K(L) \longrightarrow \mathring{\mathcal{K}}(\Xi)$, $\gamma(c) = \Xi \setminus c^{\uparrow}$ is a poset isomorphism.

[1] Observe that this is in general very different from the coarse lower topology of Ξ: Think of the case when L is distributive and $Y := \mathrm{Irr}(L)$; then $S = \emptyset$ and $\Xi := \mathrm{Irr}(L)$ is the Stone-Grätzer dual of $K(L)$.

6. Functoriality

(1) The construction of $\mathrm{Aux}(\mathrm{Ideals}(J) \setminus \{1\})$ is functorial (with values in $\mathtt{Domains}^{\mathrm{opp}}$), but we have seen that this does – a priori – not generalize classical dualities. It needs to be seen where this goes.

6. Functoriality

(1) The construction of $\mathrm{Aux}(\mathrm{Ideals}(J) \setminus \{1\})$ is functorial (with values in $\mathtt{Domains}^{\mathrm{opp}}$), but we have seen that this does – a priori – not generalize classical dualities. It needs to be seen where this goes.

(2) On the other hand the construction in the main theorem is not functorial if we take for Y the set of irreducible elements: the key problem with these elements is that they are not preserved under morphisms.

6. Functoriality

(1) The construction of $\mathrm{Aux}(\mathrm{Ideals}(J) \setminus \{1\})$ is functorial (with values in $\mathtt{Domains}^{\mathrm{opp}}$), but we have seen that this does – a priori – not generalize classical dualities. It needs to be seen where this goes.

(2) On the other hand the construction in the main theorem is not functorial if we take for Y the set of irreducible elements: the key problem with these elements is that they are not preserved under morphisms.

Hence we need a good choice of Y, which is indicated on the next page.

6. Functoriality

(1) The construction of $\mathrm{Aux}(\mathrm{Ideals}(J) \setminus \{1\})$ is functorial (with values in $\texttt{Domains}^{\mathrm{opp}}$), but we have seen that this does – a priori – not generalize classical dualities. It needs to be seen where this goes.

(2) On the other hand the construction in the main theorem is not functorial if we take for Y the set of irreducible elements: the key problem with these elements is that they are not preserved under morphisms.

Hence we need a good choice of Y, which is indicated on the next page.

Caveat: We still need to restrict morphisms in the category of semilattices. We may take the 'exact' ones (Rick Ball's name [Bal84] for a gadget derived from the construction of the injective hull of J due to Bruns and Lakser [BL70; HK71]).

$\curvearrowright$

7. Adequate choice of the set Y in the main theorem

J

$\curvearrowright$

7. Adequate choice of the set Y in the main theorem

$$\begin{tikzcd}
\tau^{\mathrm{L}}(J) \\
J \arrow[u, "s \mapsto J \setminus s^{\uparrow}"]
\end{tikzcd}$$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & \\
& (L,\tau^{\ell}(L)) \arrow[ul, dashed, hook'] \\
J \arrow[uu, "s\mapsto J\setminus s^{\uparrow}"] \arrow[ur, "s\mapsto s^{\downarrow}"'] &
\end{tikzcd}
\qquad L=\{\text{ideals of } J\}
$$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & & \\
& & (L, \tau^{\ell}(L)) \arrow[ull, hook, dashed] \\
& K(L) \arrow[ur, hook] & \\
J \arrow[uuu, "s \mapsto J \setminus s^{\uparrow}"] \arrow[ur, "\cong", "s \mapsto s^{\downarrow}"'] & &
\end{tikzcd}
$$

$L = \{\text{ideals of } J\}$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & & \\
& & (L,\tau^{\ell}(L)) \arrow[llu, hook, dashed] \\
& K(L) \arrow[ur, hook] & \\
J \arrow[uuu, "s\mapsto J\setminus s^{\uparrow}"] \arrow[ur, "\cong", "s\mapsto s^{\downarrow}"'] & & {\color{red}X :=}\,\mathrm{Irr}(L) \arrow[uu, hook]
\end{tikzcd}
$$

$L = \{\text{ideals of } J\}$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & & \\
& & (L,\tau^{\ell}(L)) \arrow[ull, dashed, hook] \\
& K(L) \arrow[ur, hook] & \\
J \arrow[uuu, "s\mapsto J\setminus s^{\uparrow}"] \arrow[ur, "\cong", "s\mapsto s^{\downarrow}"'] & & {\color{red}X :=}\mathrm{Irr}(L) \arrow[uu, hook] \\
& & \mathrm{LocCl}(X) \arrow[u, hook, "\text{sobrification}"]
\end{tikzcd}
$$

$L = \{\text{ideals of } J\}$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & & \\
\mathrm{Inj}(J) \arrow[u, hook] & & (L, \tau^{\ell}(L)) \arrow[ull, hook, dashed] \\
& K(L) \arrow[ur, hook] & \\
J \arrow[uu, "s \mapsto J \setminus s^{\uparrow}"] \arrow[ur, "s \mapsto s^{\downarrow}"', "\cong"] & & {\color{red} X :=} \mathrm{Irr}(L) \arrow[uu, hook] \\
& & \mathrm{LocCl}(X) \arrow[u, hook, "\text{sobrification}"]
\end{tikzcd}
$$

$L = \{\text{ideals of } J\}$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & & \\
\mathrm{Inj}(J) \arrow[u, hook] & & (L, \tau^{\ell}(L)) \arrow[ull, hook, dashed] \\
\mathrm{d}\ell(J) \arrow[u, hook] & K(L) \arrow[ur, hook] & \\
J \arrow[u, "s \mapsto J \setminus s^{\uparrow}"] \arrow[ur, "s \mapsto s^{\downarrow}"', "\cong"] & & X := \mathrm{Irr}(L) \arrow[uu, hook] \\
& & \mathrm{LocCl}(X) \arrow[u, hook, "\text{sobrification}"]
\end{tikzcd}
$$

$L = \{\text{ideals of } J\}$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & & & \\
\mathrm{Inj}(J) \arrow[u, hook] & & (L, \tau^{\ell}(L)) \arrow[ull, hook, dashed] & L = \{\text{ideals of } J\} \\
\mathrm{d}\ell(J) \arrow[u, hook] & K(L) \arrow[ur, hook] & & \mathrm{Spec}(\mathrm{d}\ell(J)) {\color{red}=: Y} \arrow[ul, "\bigvee \mathfrak{p} \leftarrow\!\shortmid \mathfrak{p}"'] \\
J \arrow[u, "s \mapsto J \setminus s^{\uparrow}"] \arrow[ur, "s \mapsto s^{\downarrow}"', "\cong"] & & {\color{red}X :=} \mathrm{Irr}(L) \arrow[uu, hook] & \\
& & \mathrm{LocCl}(X) \arrow[u, hook, "\text{sobrification}"] &
\end{tikzcd}
$$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & & & \\
\mathrm{Inj}(J) \arrow[u, hook] & & (L, \tau^{\ell}(L)) \arrow[ull, hook, dashed] & L = \{\text{ideals of } J\} \\
\mathrm{d}\ell(J) \arrow[u, hook] & K(L) \arrow[ur, hook] & \{\bigvee \mathfrak{p} \mid \mathfrak{p} \in Y\} \arrow[u, hook] \arrow[r, leftrightarrow, "\bigvee \mathfrak{p} \leftarrow\!\shortmid \mathfrak{p}", "\cong"'] & \mathrm{Spec}(\mathrm{d}\ell(J)) =: Y \\
J \arrow[u, "s \mapsto J \setminus s^{\uparrow}"] \arrow[ur, "\cong", "s \mapsto s^{\downarrow}"'] & & X := \mathrm{Irr}(L) \arrow[u, hook] & \mathrm{LocCl}(Y) \arrow[u, hook, "\text{sobrification}"] \\
& & \mathrm{LocCl}(X) \arrow[u, hook, "\text{sobrification}"] \arrow[ur, tail] &
\end{tikzcd}
$$

7. Adequate choice of the set Y in the main theorem

$$
\begin{tikzcd}
\tau^{\mathrm{L}}(J) & & \tau^{\ell}(L) \arrow[r, two heads] & \mathcal{O}(X) \\
\mathrm{Inj}(J) \arrow[u, hook] & & (L, \tau^{\ell}(L)) \arrow[ull, dashed, hook] \arrow[u, tail, "a \mapsto L \setminus a^{\uparrow}"] \arrow[ur, tail] & L = \{\text{ideals of } J\} \\
\mathrm{d}\ell(J) \arrow[u, hook] & K(L) \arrow[ur, hook] & \{\bigvee \mathfrak{p} \mid \mathfrak{p} \in Y\} \arrow[u, hook] \arrow[r, leftrightarrow, "\bigvee \mathfrak{p} \mapsfrom \mathfrak{p}", "\cong"'] & \mathrm{Spec}(\mathrm{d}\ell(J)) =: Y \\
J \arrow[u, "s \mapsto J \setminus s^{\uparrow}"] \arrow[ur, "\cong", "s \mapsto s^{\downarrow}"'] & & X := \mathrm{Irr}(L) \arrow[u, hook] & \mathrm{LocCl}(Y) \arrow[u, hook, "\text{sobrification}"] \\
& & \mathrm{LocCl}(X) \arrow[u, hook, "\text{sobrification}"] \arrow[ur, tail] &
\end{tikzcd}
$$

[Bal84] Richard N. Ball. "Distributive Cauchy lattices". In: *Algebra Universalis* 18.2 (1984), pp. 134–174 (cit. on pp. 59–62).

[BCM25] Guram Bezhanishvili, Luca Carai, and Patrick Morandi. *Duality Theory for Bounded Lattices: A Comparative Study*, https://arxiv.org/abs/2502.21307. 2025 (cit. on pp. 8–14).

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*. Vol. 36. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Translated from the 1987 French original, Revised by the authors. Springer-Verlag, Berlin, 1998, pp. x+430 (cit. on pp. 19–23).

[BL70] G. Bruns and H. Lakser. "Injective hulls of semilattices". In: *Canad. Math. Bull.* 13 (1970), pp. 115–118 (cit. on pp. 59–62).

[CG20] Sergio A. Celani and Luciano J. González. "A categorical duality for semilattices and lattices". In: *Appl. Categ. Structures* 28.5 (2020), pp. 853–875 (cit. on pp. 8–14).

References 2

[DST19] Max Dickmann, Niels Schwartz, and Marcus Tressl. *Spectral spaces*. Vol. 35. New Mathematical Monographs. Cambridge University Press, Cambridge, 2019, pp. xvii+633 (cit. on pp. 8–14).

[GG24] Mai Gehrke and Sam van Gool. *Topological duality for distributive lattices. Theory and applications*. English. Vol. 61. Camb. Tracts Theor. Comput. Sci. Cambridge: Cambridge University Press, 2024.

[Gie+03] G. Gierz et al. *Continuous lattices and domains*. Vol. 93. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2003, pp. xxxvi+591.

[Gou13] Jean Goubault-Larrecq. *Non-Hausdorff topology and domain theory*. Vol. 22. New Mathematical Monographs. [On the cover: Selected topics in point-set topology]. Cambridge University Press, Cambridge, 2013, pp. vi+491.

[Grä98] George Grätzer. *General lattice theory*. Second. New appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung and R. Wille. Birkhäuser Verlag, Basel, 1998, pp. xx+663 (cit. on pp. 8–14).

[HK71] A. Horn and N. Kimura. "The category of semilattices". In: *Algebra Universalis* 1.1 (1971), pp. 26–38 (cit. on pp. 59–62).

[HMS74] Karl Heinrich Hofmann, Michael Mislove, and Albert Stralka. *The Pontryagin duality of compact* O*-dimensional semilattices and its applications*. Lecture Notes in Mathematics, Vol. 396. Springer-Verlag, Berlin-New York, 1974, xvi+122pp.

[Hoc69] M. Hochster. "Prime ideal structure in commutative rings". In: *Trans. Amer. Math. Soc.* 142 (1969), pp. 43–60 (cit. on pp. 8–14).

[Hod93] Wilfrid Hodges. *Model theory*. Vol. 42. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, 1993, pp. xiv+772 (cit. on pp. 19–23).

[MJ14] M. Andrew Moshier and Peter Jipsen. "Topological duality and lattice expansions, I: A topological construction of canonical extensions". In: *Algebra Universalis* 71.2 (2014), pp. 109–126 (cit. on pp. 8–14).

[Urq78] Alasdair Urquhart. "A topological representation theory for lattices". In: *Algebra Universalis* 8.1 (1978), pp. 45–58.