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RESIDUATED LATTICES:

An algebraic glimpse at logics without contraction

(preliminary report)

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INTRODUCTION

We set out to present some recent developments in the theory of residuated lattices. As such, our task is entirely algebraic in nature, and indeed algebraic methods will be used throughout the work. However, the initial impulse for the enterprise came from without. It is their connection with logics without contraction that spurred us to investigate residuated lattices closer. From this perspective, our work falls within the area of algebraic logic with respect to its method, and substructural logics with respect to its content.

By *substructural logics* we mean, as usual, logics which, when formulated as Gentzen-style systems, lack some of the traditional trio of *structural rules*, namely: *contraction, weakening,* or *exchange* (here we understand "some" in the broad sense that includes "none"; the reasons will become clear in the next sentence). Representatives of substructural logics include Lambek calculus, linear logic, fuzzy logic, various many-valued logics, relevant logic, and, as borderline cases, intuitionism, intermediate logics, and classical logic. All of these have been subject to quite a lot of investigation, and many well-developed theories about them exist. When it comes to cross-comparisons, however, the outlook is not as bright as, for instance, in modal logic, where the general acceptance of a common semantical framework, namely, frame semantics, has facilitated communication between different "islands". In substructural area, no such common framework has yet emerged, and some developments may suggest that a contrary trend manifests itself. We believe that algebraic methods can provide a unifying semantic tool for substructural logics, and will try to demonstrate it in the present work. Thus, as the observant reader will notice, there is also a persuasive aspect to it.

Coming back to the mundane task of defining things, by *logics without contraction* we mean, roughly, these substructural logics that (may) lack contraction, but have both exchange and weakening. The smallest such logic is called $\mathbf{FL_{ew}}$, for **F**-ull **L**-ambek calculus with **e**-xchange and **w**-eakening. Having $\mathbf{FL_{ew}}$ at hand, we define (the class of) logics without contraction as (the class of all) extensions of $\mathbf{FL_{ew}}$. Logics without contraction, as defined here, admit a strong form of "algebraisation" (see Blok, Pigozzi [9] for details; for the present purposes it may be thought of in terms of Lindenbaum-Tarski algebras), and the class of algebras resulting from the process is the variety of *residuated lattices.* At this point, however, it is our duty to warn the reader that we have just made a terminological decision that is not universally followed. As far as we can tell, the term residuated lattices was first used in 1938 by Ward and Dilworth in [55], and referred to what we call residuated lattices here, or at least something very close to that. Nevertheless, there are good reasons (cf. Blyth, Janowitz [13]) for using the term to denote a much broader class of algebras, and indeed, some use it that way (cf. e.g., Blount [12]). A more restricted usage, albeit still more general than what we propose, can be found in Birkhoff [2], and more recently, e.g., in Jipsen [31]. We have no quarrel with such usages, indeed we would be willing to adjust our terminology to the standard, if some such emerges. In our present idiolect, however, the term residuated lattices will refer to bounded lattices with two additional binary operations: '$\cdot$' (known as *multiplication*, or *fusion*) and '$\to$' (*residuation*, or *implication*), such that the multiplication turns the universe into a commutative monoid, where unit coincides with the top of the lattice, and residuation being what the name suggests (cf. [13]), with respect to multiplication. These indeed form a variety, call it $\mathcal{R}$, such that the study of *logics over* $\mathbf{FL_{ew}}$ amounts, or at least reduces to a large degree, to investigating subvarieties of $\mathcal{R}$.

1. BASICS

Algebraic trivia

Our terminology and notation will generally follow that of McKenzie, McNulty, Taylor [40]. We also refer the reader to Burris, Sankappanavar [14], especially for topics not (yet) covered in [40]. Since we envisage our "target reader" as a logician rather then an algebraist, below we list some of the most frequently used algebraic concepts.

Boldface capitals stand for algebras, and italicised capitals for their universes. **Con A** stands for the *congruence lattice* of an algebra $\mathbf{A}$, whereas Con $\mathbf{A}$ denotes the bare set of all congruences of $\mathbf{A}$. Further, $\mathbf{Cg}^{\mathbf{A}}(X)$ and $\mathbf{Sg}^{\mathbf{A}}(X)$ stand for the congruence generated by the set $X \subseteq A$ and the subuniverse generated by $X \subseteq A$, respectively. We will write $\mathbf{Cg}^{\mathbf{A}}(a, b)$ instead of $\mathbf{Cg}^{\mathbf{A}}(\{a, b\})$ in the case of *principal* congruences, i.e., congruences generated by a single pair of elements. A congruence is called *compact,* or finitely generated iff it is a join of finitely many principal congruences.

If **Con A** is a distributive lattice, the algebra $\mathbf{A}$ is called *congruence distributive;* if for $\alpha, \beta \in$ Con $\mathbf{A}$ we have $\alpha \circ \beta = \beta \circ \alpha$, $\mathbf{A}$ is called *congruence permutable;* when this is the case, we also have $\alpha \vee \beta = \alpha \circ \beta$. An algebra is *arithmetical* iff it is both congruence distributive and permutable. If for any $\alpha, \beta \in$ Con $\mathbf{A}$ and any $x \in A$ we have: $x/\alpha = x/\beta$ (where x/ϕ denotes the ϕ-congruence class of x) implies $\alpha = \beta$, then **Con A** is called *regular.* If any of the above properties is true of all algebras in a class $\mathcal{K}$, the appropriate notion is applied to the whole class.

A class $\mathcal{K}$ of algebras has the *congruence extension property* (CEP) iff for any $\mathbf{A} \subseteq \mathbf{B} \in \mathcal{K}$ and any $\Theta \in$ Con $\mathbf{A}$, there is a $\Phi \in$ Con $\mathbf{B}$ such that Θ is the restriction of Φ to $A \times A$.

As usual, $I(\mathcal{K})$, $H(\mathcal{K})$, $S(\mathcal{K})$, $P(\mathcal{K})$, $P_S(\mathcal{K})$, $P_U(\mathcal{K})$, will denote the classes of: all isomorphic images, homomorphic images, subalgebras, direct products (which always contains the trivial algebra, as the product with empty set of coordinates), subdirect products, and ultraproducts of the algebras from $\mathcal{K}$.

Algebras are classified into varieties (and quasivarieties). A class $\mathcal{K}$ of algebras is a variety (quasivariety) iff $\mathcal{K}$ is closed under homomorphic images, subalgebras, and direct products (isomorphic images, subalgebras, direct products, and ultraproducts) iff $\mathcal{K} = HSP(\mathcal{K})$ $\big(\mathcal{K} = ISPP_U(\mathcal{K})\big)$. By a fundamental theorem of Bikhoff's, a class $\mathcal{K}$ of algebras is a variety iff $\mathcal{K}$ is an equational class, i.e., $\mathcal{K}$ consists of precisely these algebras that verify all equations (identities) that logically follow from a given set Γ, which is often said to *axiomatise* the variety $\mathcal{K}$. By an analogous theorem, a class $\mathcal{Q}$ of algebras is a quasivariety iff $\mathcal{K}$ is a quasi-equational class, i.e., $\mathcal{K}$ consists of precisely these algebras that satisfy all quasi-equations (quasi-identities) that logically follow from a given set Δ. By a quasiidentity, we mean a first-order open formula of the form $\tau_0 \wedge \ldots \wedge \tau_{k-1} \Rightarrow \sigma$, where $\tau_0, \ldots, \tau_{k-1}, \sigma$ are identities. Quasi-identities are the same thing as *universal Horn clauses.*

For a class of algebras $\mathcal{K}$ of the same type, $\mathcal{V}(\mathcal{K})$ and $\mathcal{Q}(\mathcal{K})$ will stand, respectively, for the variety and the quasivariety generated by $\mathcal{K}$. Subvarieties and subquasivarieties of a variety $\mathcal{V}$ form complete lattices under the operations of set-theoretical intersection and *(quasi)varietal closure* of union (i.e., smallest (quasi)variety containing union). We will write $L^q(\mathcal{V})$ for the lattice of subquasivarieties of $\mathcal{V}$, and $L^v(\mathcal{W})$ for the lattice of subvarieties of $\mathcal{V}$.

Certain algebras serve as building-blocks for others. To make this a little more precise, let $\mathbf{A}$ and $\mathbf{B}_i$ for each $i \in I$ be algebras of the same type. A *subdirect representation* of $\mathbf{A}$ with factors $\mathbf{B}_i$ is an embedding $f : \mathbf{A} \to \prod_{i\in I} \mathbf{B}_i$ such that each f_i defined by $f_i = \pi_i \circ f$ is onto $\mathbf{B}_i$ for each $i \in I$. Here, π_i denotes the i-th projection. Such an $\mathbf{A}$ is also called a *subdirect product* of $\mathbf{B}_i$. The indefinite article is appropriate: subdirect representations (products) are not unique. An algebra $\mathbf{A}$ is *subdirectly irreducible* (si) iff it is non-trivial (here we follow rather [40] than [14]) and for any subdirect representation $f : \mathbf{A} \to \prod_{i\in I} \mathbf{B}_i$, there exists a j such that f_j is an isomorphism of $\mathbf{A}$ onto $\mathbf{B}_j$. An algebra is si iff its congruence lattice has a unique atom, i.e., unique smallest non-zero congruence, called the *monolith.* A fundamental *subdirect representation theorem* of Birkhoff's says that every algebra has a subdirect representation with si factors.

Two other important types of building-blocks are: *simple* algebras, i.e., these that have two-element congruence lattices; and *directly indecomposable,* i.e., these that cannot be nontrivially represented as direct products. Clearly *simple* implies *si* implies *directly indecomposable*; neither of the converse implications holds in general.

Sometimes we will write $\mathcal{V}_{SI}(\mathcal{K})$ for the class of all si algebras from $\mathcal{V}(\mathcal{K})$, and $\mathcal{V}_S(\mathcal{K})$ for the class of all simple algebras from $\mathcal{V}(\mathcal{K})$.

Since all varieties we will be dealing with are congruence distributive, of frequent use to us will be a theorem of Jónsson's (cf. [32]), frequently referred to as *Jónsson Lemma*, stating that, for a congruence distributive variety $\mathcal{V}$ generated by a class $\mathcal{K}$, we have: $\mathcal{V}_{SI}(\mathcal{K}) \subseteq HSP_U(\mathcal{K})$.

A quite well-behaved subclass of congruence distributive varieties are varieties with *(equationally) definable principal congruences* (cf. [6], [5], [7], []). Among these, an even smaller subclass of *discriminator* varieties (cf. [56]) is very well-behaved. We define both concepts below.

A variety $\mathcal{V}$ has *definable principal congruences* (DPC) iff there is a formula $\phi(x, y, z, t)$ in the first-order language of $\mathcal{V}$ (with no other free variables than the ones displayed) such that, for any $a, b, c, d \in \mathbf{A} \in \mathcal{V}$, we have: $(c, d) \in \Theta(a, b)$ iff $\mathbf{A} \models \phi(a, b, c, d)$. If $\phi(x, y, z, t)$ can be taken to be a conjunction (i.e., a finite set) of equations, then $\mathcal{V}$ has *equationally definable principal congruences* (EDPC).

The *ternary discriminator* is a function t defined by:

$$t(x, y, z) = \begin{cases} x, & \text{for } x \neq y \\ z, & \text{for } x = y. \end{cases}$$

A *discriminator variety* is a variety $\mathcal{V}$, such that the ternary discriminator is a term-operation on every si algebra in $\mathcal{V}$.

One final remark on notation. It is customary to distinguish between the logical language (connectives, formulas, and the like) and the algebraic language (operation symbols, terms, etc.). We will flout this custom and use one and the same language for both purposes. This is justified by the simple observation that connectives and formulas of a logic can be thought of as operations on and elements of the absolutely free algebra (the algebra of terms) of the appropriate type. However, in our metalanguage of discourse we will retain both the usual logical terminology and the usual algebraic terminology, thus, we will sometimes refer to one and the same object (say, an element of an absolutely free algebra) by different names (formula, term).

Logics without contraction

We begin with formally defining the sequent calculus $\mathbf{FL_{ew}}$: our "basic" logic without contraction. As we have seen, roughly, $\mathbf{FL_{ew}}$ is the system obtained from Gentzen sequent calculus **LJ** for the intuitionistic logic by deleting the contraction rule. The language of $\mathbf{FL_{ew}}$ consists of a constant 0, and connectives $\wedge, \vee, \cdot$ and $\to$. Negation and the "truth" constant 1, are defined by $\neg\alpha = \alpha \to 0$ and $1 = \neg 0$. It will sometimes be useful to abbreviate $(\alpha \to \beta) \wedge (\beta \to \alpha)$ to $\alpha \leftrightarrow \beta$. A *sequent* is a list of formulas of the form $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$, with $m \geq 0$. The system $\mathbf{FL_{ew}}$ consists of initial sequents:

$$\alpha \Rightarrow \alpha, \qquad \text{and} \qquad 0, \Gamma \Rightarrow \gamma;$$

and rules of inference:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma} \text{ (cut)}$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma} \text{ (exchange)} \qquad \frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \text{ (weakening)}$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \ (\Rightarrow\to) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \to \beta, \Gamma, \Delta \Rightarrow \gamma} \ (\to\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 2)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee \beta, \Gamma \Rightarrow \gamma} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma} \ (\wedge 1 \Rightarrow) \qquad \frac{\beta, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma} \ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} \ (\Rightarrow \cdot) \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha \cdot \beta, \Gamma \Rightarrow \gamma} \ (\cdot \Rightarrow).$$

Provability of a given sequent is defined in the usual way. We say that a formula α is provable (in $\mathbf{FL_{ew}}$) when the sequent $\Rightarrow \alpha$ is provable (in $\mathbf{FL_{ew}}$). The system $\mathbf{FL_{ew}}$ enjoys cut elimination property, from which decidability and Craig interpolation can be derived (all proved in Ono, Komori [43], where also Hilbert-style formulation and Kripke-type semantics for $\mathbf{FL_{ew}}$ were investigated). For more on $\mathbf{FL_{ew}}$ and related systems, see e.g., Došen [19], [20], Ono [43], [46], [47].

By a *logic*, we mean any set of formulas closed under substitution and *modus ponens*. A *logic over* $\mathbf{FL_{ew}}$ (or a *logic without contraction*, or simply a *logic*, when no confusion arises) is then any logic that includes all formulas provable in (or, in other words, all *theorems* of) $\mathbf{FL_{ew}}$. Although *deductive systems*, of which Gentzen-style systems are a species, are generally different than logics (for instance, $\mathbf{FL_{ew}}$ with cut is a different deductive system than cut-free $\mathbf{FL_{ew}}$, but, by cut elimination theorem, the logics determined by these two, i.e., their sets of provable formulas, are identical), we will often identify a deductive system with the set of all formulas provable in it (again, if no confusion arises). Thus, $\mathbf{FL_{ew}}$ denotes not only the sequent calculus introduced above, but also the set of all formulas provable in it. We will commit this abuse of terminology systematically, but with two traditional exceptions. Namely, **LK** and **LJ** will always stand for Gentzen-style deductive systems, whose sets of theorems are, respectively, **Cl** (classical logic) and **Int** (intuitionistic logic).

It is easy to see that adding to $\mathbf{FL_{ew}}$ the following *contraction rule:*

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \text{ (contraction)}$$

yields a (cut-free) sequent calculus for **Int**. For any sequent of the form $\alpha \cdot \beta \Rightarrow \alpha \wedge \beta$ is provable in $\mathbf{FL_{ew}}$, and any sequent of the form $\alpha \wedge \beta \Rightarrow \alpha \cdot \beta$ is provable in $\mathbf{FL_{ec}}$ (where $\mathbf{FL_{ec}}$ arises from $\mathbf{FL_{ew}}$ by removing weakening and adding contraction). Thus, $\mathbf{FL_{ecw}} = \mathbf{LJ}$ as a deductive system, and $\mathbf{FL_{ecw}} = \mathbf{Int}$ as a logic. Incidentally, it shows that $\cdot$ is equivalent to $\wedge$ in **LJ**.

There are two subclasses of logics over $\mathbf{FL_{ew}}$ which have been very comprehensively investigated. One is the class of intermediate logics (or, superintuitionistic logics), i.e., logics between **Int** and **Cl**, and the other is the class of Łukasiewicz many-valued logics.

The class of logics over $\mathbf{FL_{ew}}$ is ordered by the set inclusion $\subseteq$. Of course, $\mathbf{FL_{ew}}$ is the smallest logic among them and the set of all formulas (the *inconsistent* logic) is the greatest one. Among *consistent* logics over $\mathbf{FL_{ew}}$ there also is the greatest, namely **Cl**.

Residuated lattices

Now we have come to *residuated lattices*, the algebraic counterpart of logics without contraction. Residuated lattices have been investigated since the late 30s, by, for instance, Krull [38] (his work is really an even earlier precursor), Dilworth [18], Ward and Dilworth [55], Ward [54], then (much later, in the 70s) Balbes and Dwinger [1], and Pavelka [48]. With Idziak [29], proving that the class of residuated lattices is an arithmetical variety, the modern era begins.

Residuated lattices have been known under extraordinarily many names, they have been called *BCK*-

lattices in [29], *full BCK-algebras* in [43], FL_{ew}*-algebras* in [47], and *integral, residuated, commutative l-monoids* in [28].

Definition 1.1. *An algebra* $\mathbf{A} = \langle A; \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ *is a* residuated lattice *if:*

(1) $\langle A, \wedge, \vee, 0, 1\rangle$ *is a bounded lattice with the greatest element 1 and the least 0,*

(2) $\langle A, \cdot, 1\rangle$ *is a commutative monoid,*

(3) *for* $x, y \in A$, $\quad x \cdot y \leq z$ *if and only if* $x \leq y \rightarrow z$.

The relation between the multiplication $\cdot$ and its *residual* $\rightarrow$ expressed by Definition 1(3) is a particular case of the *law of residuation* (cf. [13]). Namely, let $\mathbf{P}$ and $\mathbf{Q}$ be posets, and $f : P \longrightarrow Q$ any map. Then f is residuated iff there is a map $g : Q \longrightarrow P$, such that for any $p \in P$, $q \in Q$, we have: $f(p) \leq q$ iff $p \leq g(q)$. This is also expressed by saying that f and g form a *residuated pair.* Now, setting $\mathbf{P} = \mathbf{Q} = \mathbf{A}$, and defining, for any $a \in A$, two maps f_a and g_a, by putting $f_a(x) = x \cdot a$ and $g_a(x) = a \rightarrow x$, we see that $x \cdot a = f_a(x) \leq y$ iff $x \leq g_a(y) = a \rightarrow y$. Thus, f_a and g_a form a residuated pair.

Proposition 1.1. *The class* $\mathcal{R}$ *of all residuated lattices is an arithmetical variety.*

Proof. See [29] for a detailed proof. One of the equational axiomatisations of $\mathcal{R}$ can be:

(L) equations axiomatising the variety of bounded lattices,

(M) equations axiomatising the variety of commutative monoids,

(R1) $xy \rightarrow z = x \rightarrow (y \rightarrow z)$,

(R2) $(x \rightarrow y)x \leq y \quad$ (i.e., $(x \rightarrow y)x \wedge y = (x \rightarrow y)x$),

(R3) $(x \wedge y) \rightarrow y = 1$.

Proposition 1.2. *Let* $\mathbf{A}$ *be any residuated lattice,* $a, b, c \in A$, *and* $Z \subseteq A$. *The following hold:*

(i) $a \leq b$ *implies* $a \cdot c \leq b \cdot c$;

(ii) $a \cdot b \leq a$ *and thus* $a \cdot b \leq a \wedge b$;

(iii) *if* $\bigvee Z$ *exists, then* $a \cdot \bigvee Z = \bigvee\{a \cdot z : z \in Z\}$;

(vi) *if* $\bigvee Z$ *exists, then* $\bigvee Z \rightarrow a = \bigwedge\{z \rightarrow a : z \in Z\}$;

(v) *if* $\bigwedge Z$ *exists, then* $a \rightarrow \bigwedge Z = \bigwedge\{a \rightarrow z : z \in Z\}$;

(vi) $1 \rightarrow a = a$,

(vii) $a \leq b$ *iff* $a \rightarrow b = 1$;

(viii) $a \rightarrow b = \max\{z : z \cdot a \leq b\}$.

Working out the proofs is a good exercise in arithmetic of residuated lattices. Clearly, the following finite distributivity:

(iii′) $(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z)$,

follows from (iii) above. It is also clear that (ii) and (i) follow from (iii). However, neither (iii′) nor even (iii) can force the existence of the residual; however, if $\mathbf{A}$ is complete as an upper semilattice (i.e., $\bigvee Z$ exists for all $Z \subseteq A$), then (iii) suffices to guarantee that a (unique) residual exists. More precisely, let $\mathbf{A}$ be an algebra of the appropriate type satisfying conditions (1) and (2) of residuated lattices; moreover, let $\mathbf{A}$ be complete as an upper semilattice and satisfy (iii). Then, $\mathbf{A}$ is a reduct of a residuated lattice. To see that, take $\bigvee\{x : xy \leq z\}$ for $y \to z$. This exists by completeness, and by (iii) it is equal to $\max\{x : xy \leq z\}$. Then, clearly, $xy \leq z$ implies $x \leq y \to z$. For the converse, suppose $x \leq y \to z$. By definition, $(y \to z)y \leq z$. Now, by (i), which in turn follows from (ii), we get $xy \leq (y \to z)y$. This yields $xy \leq z$ as needed. In particular, if $\mathbf{A}$ is finite and satisfies (1), (2), and (iii$'$), then $\mathbf{A}$ is a reduct of a residuated lattice.

Of some importance for the theory of residuated lattices is the fact that their congruences behave rather well. To make this precise, we will need a definition.

Definition 1.2. *A nonempty subset F of a residuated lattice $\mathbf{A}$ is a* congruence filter *if for $x, y \in A$ it satisfies:*

(1) *$x \leq y$ and $x \in F$ imply $y \in F$,*

(2) *$x, y \in F$ implies $x \cdot y \in F$.*

Note that any congruence filter F is upward closed and satisfies the lattice filter condition: $x, y \in F$ implies $x \wedge y \in F$. Thus, every congruence filter is a filter in the lattice-theoretic sense, although the converse does not hold. Despite that, we adopt the convention that "filter" means "congruence filter" unless explicitly specified otherwise. It is easy to see that a nonempty subset F of a residuated lattice is a filter if and only if it satisfies (a) $1 \in F$ and (b) $x, x \to y \in F$ implies $y \in F$. Let S be a nonempty subset of a residuated lattice $\mathbf{A}$. Then the set $\{x : a_1 \cdots a_k \leq x,$ for some $a_1, \ldots, a_k \in S\}$ is a filter, called the filter *generated* by S. In particular, the filter generated by a singleton set $\{a\}$ for an element $a \in A$ is the set $\{x : a^k \leq x,$ for some positive integer $k\}$, where a^k denotes $\underbrace{a \cdots a}_{k \text{ times}}$. The following can be easily shown.

Proposition 1.3. *Let $\mathbf{A}$ be any residuated lattice, $F \subseteq A$ a filter, and α a congruence on $\mathbf{A}$. Then, the relation θ_F defined by: $(x, y) \in \theta_F$ iff $x \to y,\ y \to x \in F$, is a congruence on $\mathbf{A}$; the subset F_α defined by: $a \in F_\alpha$ iff $(a, 1) \in \alpha$, is a filter on $\mathbf{A}$. Moreover, the natural maps associated with the above are mutually inverse and establish an isomorphism between the lattice of all filters of $\mathbf{A}$ and* $\mathbf{Con\,A}$.

It follows immediately from the proposition above, that a residuated lattice $\mathbf{A}$ is subdirectly irreducible if and only if it has the second smallest filter, i.e. the smallest filter among all filters except $\{1\}$.

The next lemma characterises "internally" si and simple residuated lattices.

Lemma 1.1. *A residuated lattice $\mathbf{A}$ is subdirectly irreducible iff there exists an element $a < 1$ such that for any $x < 1$ there exists a positive integer m for which $x^m \leq a$ holds. A residuated lattice $\mathbf{A}$ is simple iff a can be taken to be* 0.

Proposition 1.4. *In any si residuated lattice, if $x \vee y = 1$ then either $x = 1$ or $y = 1$ holds. Therefore, every si residuated lattice has at most one coatom.*

Proof. By contraposition, it suffices to show that $x, y < 1$ implies $x \vee y < 1$ in a given subdirectly irreducible residuated lattice $\mathbf{A}$. Since $\mathbf{A}$ is subdirectly irreducible, there exists $a < 1$ such that for any $z < 1$ there exists a number k satisfying $z^k \leq a$. In particular, both $x^m \leq a$ and $y^n \leq a$ hold for some positive integers m and n. Define $s = \max\,\{m, n\}$ and $t = 2s - 1$. Then, clearly $x^s \leq a$ and $y^s \leq a$ hold. Now, by the distributivity of $\cdot$ over $\vee$:

$$(x \vee y)^t = \bigvee_{i=1}^{t} x^i \cdot y^{t-i}.$$

It is easy to see that either $i \geq s$ or $t - i \geq s$. Hence, in the former case,

$$x^i \cdot y^{t-i} \leq x^i \leq x^s \leq a$$

and in the latter case,

$$x^i \cdot y^{t-i} \leq y^{t-i} \leq y^s \leq a.$$

Thus, in either case, $(x \vee y)^t \leq a$. Therefore, $x \vee y$ cannot be equal to 1.

To prove the second statement, recall that an element a of a lattice $\mathbf{L}$ with the greatest element 1 is a *coatom* if it is maximal among elements in $L - \{1\}$. If our si residuated lattice $\mathbf{A}$ had two coatoms c_0 and c_1, then by the first statement we had $c_0 = 1$ or $c_1 = 1$, in either case a contradiction.

The next lemma (proved in Kowalski [35] for BCK-algebras) characterises these si residuated lattices which have the coatom. Note the swap of quantifiers in comparison with Lemma 1.1.

Lemma 1.2. *A residuated lattice $\mathbf{A}$ has the unique coatom if and only if there exists an element $a < 1$ and a positive integer m such that $x^m \leq a$ holds for any $x < 1$.*

Proof. To show the only-if part, it is enough to take the coatom for a and 1 for m. Conversely, suppose that there exists $a(< 1)$ and $m \geq 1$ such that $x^m \leq a$ for any $x < 1$. By Lemma 1.1, it is clear that $\mathbf{A}$ is subdirectly irreducible. Now, take any such a and take also the smallest number k among such ms for this a. If $k = 1$ then it is obvious that a is the single coatom of $\mathbf{A}$. So, suppose that $k > 1$. By our assumption, there exists an element b such that $b^{k-1} \nleq a$ but $b^k \leq a$. Define d by $d = b^{k-1} \to a$. Clearly, $d < 1$. We will show that d is the single coatom of $\mathbf{A}$. Take any y such that $y < 1$ and let $z = d \vee y$. Then, $d, y \leq z$ and moreover $z < 1$ by Proposition 1.4. Since $b^k \leq a$, $b \leq b^{k-1} \to a = d \leq z$. As $z < 1$, we have $z^k \leq a$ by our assumption. Thus, $y \leq z \leq z^{k-1} \to a \leq b^{k-1} \to a = d$. Therefore, $y \leq d$. Hence, d is the coatom of $\mathbf{A}$.

Directly indecomposable residuated lattices also have quite a handy description. It was obtained for a subvariety of residuated lattices, called *product algebras* (see remarks on fuzzy logic a few pages ahead) by Cingnoli and Torrens in [16]. We generalise it here to arbitrary residuated lattices.

Let $\mathbf{L}$ be a bounded lattice. Recall that an element a of L is called *complemented* iff there is an element $b \in A$, with $a \vee b = 1$ and $a \wedge b = 0$. If such a b exists it is called a *complement* of a. We will denote the set of complemented elements in $\mathbf{L}$ by $C(\mathbf{L})$. Complements are generally not unique, unless the lattice is distributive. In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.

Lemma 1.3. *Let* $\mathbf{A}$ *be any residuated lattice, and let* $a \in A$ *have a complement* $b \in A$*. Then, the following hold:*

(i) *if* c *is a complement of* a *in* A*, then* $c = b$*;*

(ii) $\neg a = b$ *and* $\neg b = a$*;*

(iii) $a^2 = a$*.*

Proof. Since b and c are both complements of a, we have: $a \vee b = 1 = a \vee c$. Then, $c = c \cdot 1 = c(a \vee b) = ca \vee cb = cb$, since $ca \leq c \wedge a = 0$. Similarly, $b = b(a \vee c) = ba \vee bc = bc = cb$. Thus, $b = c$ as needed.

For the second statement, observe that $\neg a = a \to 0 \geq b$, for $ab \leq a \wedge b = 0$. Then take: $\neg a = (\neg a) \cdot 1 = (\neg a)(a \vee b) = (\neg a)a \vee (\neg a)b = (\neg a)b$. Thus, we have: $(\neg a)b = \neg a \geq \neg a \wedge b$, but on the other hand, $(\neg a)b \leq \neg a \wedge b$. Hence, $\neg a = (\neg a)b = \neg a \wedge b$, which means that $b \geq \neg a$. Altogether, we obtain $\neg a = b$, as required.

For the third claim, take: $a = a(a \vee b) = a^2 \vee ab = a^2$.

For any algebra $\mathbf{A}$, and a pair of congruences α, β on $\mathbf{A}$, we say that α and β are *complementary factor congruences* iff $\alpha \vee \beta = 1$, $\alpha \wedge \beta = 0$, and $\alpha \circ \beta = \beta \circ \alpha$. In other words, α and β are complementary factor congruences iff they are complements of one another in the congruence lattice, and moreover they permute.

Lemma 1.4. *If* α *and* β *are complementary factor congruences on* $\mathbf{A}$*, then* $\mathbf{A} \cong \mathbf{A}/\alpha \times \mathbf{A}/\beta$*.*

Let us now come back to residuated lattices. Let $\mathbf{A}$ be a nontrivial residuated lattice, and $a \in A$ a complemented element. For an element b of A, let $F(b)$ stand for the congruence filter generated by b, and for a congruence filter F, let ϕ_F stand for the corresponding congruence.

Lemma 1.5. *The congruences* $\phi_{F(a)}$ *and* $\phi_{F(\neg a)}$ *form a pair of complementary factor congruences.*

Proof. Since the variety of residuated lattices is congruence permutable, we only have to show, that $\phi_{F(a)}$ and $\phi_{F(\neg a)}$ are complements. By the correspondence between filters and congruences, our task further reduces to showing that: (1) $F(a) \cap F(\neg a) = \{1\}$, and (2) $F(a) \vee F(\neg a) = A$, with the join taken in the lattice of filters.

Now, by Lemma 1.3(iii), we have that $F(a) = \{x \in A; x \geq a\}$ and $F(\neg a) = \{x \in A; x \geq \neg a\}$. As $a \vee \neg a = 1$, we obtain that $F(a) \cap F(\neg a) = \{x \in A; x \geq a\} \cap \{x \in A; x \geq \neg a\} = \{1\}$, proving (1). Then, $F = F(a) \vee F(\neg a)$ is the smallest filter containing $\{x \in A; x \geq a\} \cup \{x \in A; x \geq \neg a\}$. Since $a \cdot (\neg a) \leq a \wedge \neg a = 0$, we get that $0 \in F$. Thus $F = A$, finishing the proof.

Let $C(\mathbf{A})$ be the set of all complemented elements of a residuated lattice $\mathbf{A}$. The next theorem follows immediately from what we said above.

Proposition 1.5. *A nontrivial residuated lattice* $\mathbf{A}$ *is directly indecomposable iff* $C(\mathbf{A}) = \{0, 1\}$

Now we want to take a closer look at $C(\mathbf{A})$. Since all si algebras are directly indecomposable, we conclude by Proposition 1.5 that if $\mathbf{A}$ is si, then $C(\mathbf{A}) = \{0, 1\}$.

Lemma 1.6. *Any element a of $C(\mathbf{A})$ has $a \cdot x = a \wedge x$, for any $x \in A$.*

Proof. Let h_i $(i \in I)$ be a system of homomorphisms constituting a subdirect representation of $\mathbf{A}$, with si factors $\mathbf{A}_i$ $(i \in I)$. Take an $a \in C(\mathbf{A})$, and $x \in A$. Observe that by subdirect irreducibility of each $\mathbf{A}_i$, $h_i(a)$ can be either 0, or 1. Calculate: $h_i(ax) = h_i(a) \cdot h_i(x) = h_i(a) \wedge h_i(x) = h_i(a \wedge x)$, for all $i \in I$. Therefore, $ax = a \wedge x$, as needed.

Lemma 1.7. *If a, b belong to $C(\mathbf{A})$, so do $a \vee b$, and $a \wedge b$. Moreover, $\neg(a \vee b) = \neg a \wedge \neg b$ and $\neg(a \wedge b) = \neg a \vee \neg b$ both hold.*

Proof. First note that the identity $\neg(a \vee b) = \neg a \wedge \neg b$ (one-half of de Morgan law) holds in all residuated lattices. Thus, it will be convenient to start with showing that $a \vee b \in C(\mathbf{A})$. For whenever an element has a complement, the complement is always equal to its negation. Thus, we only have to show, that $\neg a \wedge \neg b$ is the complement of $a \vee b$.

Suppose $(\neg a \wedge \neg b) \vee a \vee b < 1$. Then, taking a subdirect representation of $\mathbf{A}$ as in the previous proof, we obtain that for at least one $i \in I$, $h_i((\neg a \wedge \neg b) \vee a \vee b) < h_i(1) = 1_i$. As h_i is a homomorphism, this yields: $(h_i(\neg a) \wedge h_i(\neg b)) \vee h_i(a) \vee h_i(b) < 1_i$. Therefore, $h_(a) = 0 = h_i(b)$ as otherwise the whole term on the left-hand side would be equal to 1. But then, $h_i(\neg a) = 1 = h_i(\neg b)$ and the term is equal to 1 anyway. Contradiction.

Suppose now $(\neg a \wedge \neg b) \wedge (a \vee b) > 0$. Reasoning as above, we get $(h_i(\neg a) \wedge h_i(\neg b)) \wedge (h_i(a) \vee h_i(b)) > 0_i$. This also leads to contradiction, for we must have $h_i(\neg a) = 1 = h_i(\neg b)$, which forces $h_i(a) = 0 = h_i(b)$.

This proves that both $a \vee b$ and $\neg a \wedge \neg b$ belong to $C(\mathbf{A})$, and they are complements of one another. Thus, $\neg(\neg a \wedge \neg b) = a \vee b$ holds. Moreover, since $\neg a, \neg b \in C(\mathbf{A})$ as well, we have $\neg(\neg\neg a \wedge \neg\neg b) = \neg a \vee \neg b$. Then, by double negation, for members of $C(\mathbf{A})$, we get $\neg(a \wedge b) = \neg a \vee \neg b$, which is exactly the other half of de Morgan law. This enables us to carry out the proof of the fact that $a \wedge b$ and $\neg a \vee \neg b$ are complements simply by dualising the previous argument.

An easy consequence of Lemma 1.7 is that $C(\mathbf{A})$ is (the universe of) a de Morgan lattice. All we need now to conclude that it is in fact a Boolean algebra is to prove distributivity.

Proposition 1.6. *The set $C(\mathbf{A})$ is the universe of a Boolean subalgebra of $\mathbf{A}$.*

Proof. We prove that for any $a, b, c \in C(\mathbf{A})$, the distributive law holds. By Lemma 1.6, and properties of residuated lattices, we have the following series of identities: $a \wedge (b \vee c) = a(b \vee c) = ab \vee ac = (a \wedge b) \vee (a \wedge c)$.

Logics and algebras

We will now focus on the connection between logics without contraction and residuated lattices. The connection will turn out to be so tight, that we will be able to give up the logical side of things entirely (well, almost) and always talk in algebraic argot. Here is how.

In the usual way, we will define the *validity* of formulas (of $\mathbf{FL_{ew}}$) in a given residuated lattice $\mathbf{A}$ as follows. Let $\mathbf{T}_{\mathcal{R}}(\overline{\omega})$ be the absolutely free algebra of the type of residuated lattices generated by the countable set $\overline{\omega}$ of *propositional variables*. Any mapping $f : \overline{\omega} \longrightarrow A$ can be extended to a homomorphism $v : \mathbf{T}_{\mathcal{R}}(\overline{\omega}) \longrightarrow \mathbf{A}$, as follows:

(a) $v(0) = 0$,

(b) $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$,

(c) $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$,

(d) $v(\alpha \cdot \beta) = v(\alpha) \cdot v(\beta)$,

(e) $v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$.

Any such v is called a *valuation*. A formula α is *true*) (or *valid*) in $\mathbf{A}$ if $v(\alpha) = 1$ holds for any valuation on $\mathbf{A}$. Algebraically, it means that the equation $\alpha = 1$ is true in $\mathbf{A}$. Conversely, if an equation $\tau = \sigma$ is true in a residuated lattice $\mathbf{A}$, then the formula $\tau \leftrightarrow \sigma$ is valid in $\mathbf{A}$.

We will use the usual notation $\mathbf{A} \models \tau = \sigma$, to express the fact that the equation $\tau = \sigma$, with τ and σ being terms, is true in $\mathbf{A}$. By our conventions, $\mathbf{A} \models \alpha$ is simply a shorthand for $\mathbf{A} \models \alpha = 1$. Occasionally we will also write $\mathbf{A} \models_v \tau = \sigma$, or $\mathbf{A} \models_v \alpha$, to mean, respectively: $v(\tau) = v(\sigma)$ in $\mathbf{A}$, and $v(\alpha) = 1$ in $\mathbf{A}$.

The set of formulas which are valid in $\mathbf{A}$ is denoted by $L(\mathbf{A})$. The set of all equations true in $\mathbf{A}$ (called the *equational theory* of $\mathbf{A}$) is denoted by $\mathrm{Id}(\mathbf{A})$. For sequents, we say that a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is valid in $\mathbf{A}$ iff the formula $(\alpha_1 \cdot \ldots \cdot \alpha_m) \Rightarrow \beta$ is valid in $\mathbf{A}$.

All these notions extend in the natural way to subclasses (subvarieties) of $\mathcal{R}$. Namely, a formula is valid in a class of residuated lattices iff it is valid in all algebras from the class; and likewise for sequents. For a subvariety $\mathcal{V}$ of $\mathcal{R}$, we will write $L(\mathcal{V})$ for the set of all formulas valid in $\mathcal{V}$, and $\mathrm{Id}(\mathcal{V})$ for the equational theory of $\mathcal{V}$. Clearly, if $\mathcal{V}$ is generated by a class $\mathcal{K}$, then $L(\mathcal{V}) = L(\mathcal{K})$. In particular, if $\mathcal{V} = \mathcal{V}(\mathbf{A})$, then $L(\mathcal{V}) = L(\mathbf{A})$. Let $\mathbf{F}_{\mathcal{V}}$ stand for the free algebra in $\mathcal{V}$, countably generated by the set $\overline{\omega}$.

Now, suppose L is a logic over $\mathbf{FL_{ew}}$. We set $\mathcal{V}(L)$ to be the class of all residuated lattices validating L. Clearly, $\mathcal{V}(L)$ is a variety. Note that the free algebras $\mathbf{F}_{\mathcal{V}(L)}$ have long been known to logicians as *Lindenbaum algebras*.

Proposition 1.7. *Let $\mathcal{V}$ be a subvariety of $\mathcal{R}$, L a logic over $\mathbf{FL_{ew}}$, and S a sequent. We have:*

(i) *S is provable in $L(\mathcal{V})$ iff S it is valid in $\mathcal{V}$,*

(ii) $L(\mathcal{V}) = \{\tau \leftrightarrow \sigma : \tau = \sigma \in \mathrm{Id}(\mathcal{V})\}$,

(iii) $\mathrm{Id}(\mathcal{V}) = \{\varphi = 1 : \varphi \in L(\mathcal{V})\}$,

(iv) $L(\mathcal{V}) = L(\mathbf{F}_{\mathcal{V}})$,

(v) $L(\mathcal{V}(L)) = L$

(vi) $\mathcal{V}(L(\mathcal{V})) = \mathcal{V}$.

Proposition 1.7 enables us to speak of varieties corresponding to logics, and the other way round. Namely, for any index set I let L_i and $\mathcal{V}_i$ $(i \in I)$ stand respectively for a family of logics and a family of varieties, such that $\mathcal{V}_i = \mathcal{V}(L_i)$ and *vice versa*. Moreover, let $\bigvee$ stand for a lattice join, be it the join in the lattice of logics, or in the lattice of varieties. It is defined as the smallest logic/variety containing all of the joinands. Let ν be a map from the $L^v(\mathcal{R})$ to the lattice of logics over $\mathbf{FL_{ew}}$, defined by $\nu(\mathcal{V}) = L(\mathcal{V})$. Let μ be a map from the lattice of logics over $\mathbf{FL_{ew}}$ to $L^v(\mathcal{R})$, defined by $\mu(L) = \mathcal{V}(L)$. The lattices above are both complete, with meet being the set-theoretical intersection, and join defined as above.

Proposition 1.8. *The maps ν and μ are mutually inverse and establish a dual complete lattice isomorphism between the lattice of logics over $\mathbf{FL_{ew}}$ and $L^v(\mathcal{R})$. In particular:*

(i) $L\big(\bigvee_{i\in I}\mathcal{V}_i\big) = \bigcap_{i\in I} L(\mathcal{V}_i)$,

(ii) $L\big(\bigcap_{i\in I}\mathcal{V}_i\big) = \bigvee_{i\in I} L(\mathcal{V}_i)$,

(iii) $\mathcal{V}\big(\bigvee_{i\in I} L_i\big) = \bigcap_{i\in I} \mathcal{V}(L_i)$,

(iv) $\mathcal{V}\big(\bigcap_{i\in I} L_i\big) = \bigvee_{i\in I} \mathcal{V}(L_i)$.

In plain language this can be expressed by saying that the lattice of logics over $\mathbf{FL_{ew}}$ is the lattice of subvarieties of $\mathcal{R}$ upside down.

Now, suppose L_i are logics over $\mathbf{FL_{ew}}$, with some index set I. Let Λ be the set of all formulas α such that there exist $j_1, \ldots, j_k \in I$ and formulas $\beta_{j_t} \in L_{j_t}$ for $1 \le t \le k$ such that the formula $(\beta_{j_1} \cdot \ldots \cdot \beta_{j_k}) \Rightarrow \alpha$ is provable in $\mathbf{FL_{ew}}$. Then, it can be easily shown that $\Lambda = \bigvee_{i\in I} L_i$. This can be used to derive the following proposition directly, however it also follows from congruence distributivity of residuated lattices via a result of Jónsson [32] and duality between the lattice of logics over $\mathbf{FL_{ew}}$ and $L^v(\mathcal{R})$.

Proposition 1.9. *In the complete lattice of all logics over $\mathbf{FL_{ew}}$ the following distributive law holds:*

$$L \wedge \bigvee_i L_i = \bigvee_i (L \wedge L_i).$$

Let L_0 and L be logics such that $L_0 \subseteq L$. Then, L is said to be *finitely axiomatised over L_0 by the axioms* $\alpha_1, \ldots, \alpha_m$, if L is the smallest logic which contains both L_0 and the set $\{\alpha_1, \ldots, \alpha_m\}$. Then, for any formula γ, $\gamma \in L$ iff there are formulas $\beta_1, \ldots, \beta_n$ (for some $n \ge 0$), each being a substitution instance of some α_k, such that the formula $(\beta_1 \cdot \ldots \cdot \beta_n) \to \gamma$ belongs to L_0. Such a logic L will be denoted by $L_0[\alpha_1, \ldots, \alpha_m]$. A logic L is *finitely axiomatisable* over L_0, if there are a finite number of axioms by which L is axiomatised over L_0. We will omit the words "over L_0" when L_0 is $\mathbf{FL_{ew}}$. It is easy to see that $L[\alpha_1, \ldots, \alpha_m] = L[\alpha_1 \cdot \ldots \cdot \alpha_m]$, with the help of the *weakening rule* of $\mathbf{FL_{ew}}$, i.e., using the fact that $(\gamma \cdot \delta) \to \gamma$ is provable in $\mathbf{FL_{ew}}$. We have:

Proposition 1.10. *Let L and L' be finitely axiomatized over L_0 by the axioms α and β, respectively. Then, logics $L \wedge L'$ and $L \vee L'$ are finitely axiomatized over L_0 respectively by axioms $\alpha \vee \beta$ and $\alpha \cdot \beta$; in the second case with the proviso that α and β have no propositional variables in common.*

Noteworthy logics in the vicinity

If you do not insist on double precision and absolute responsibility, you can say that the classes of substructural logics that have attained fame are three. Namely, in alphabetical order: fuzzy logics, linear logics, and relevant logics. Of these, as we will soon see, only fuzzy logics are logics over $\mathbf{FL_{ew}}$. Linear logics (see, e.g., Troelstra [52]) are extensions of $\mathbf{FL_e}$ (so-called *intuitionistic* linear logics). In original Girard formulation (see [25]) they were extensions of $\mathbf{FL_e}$ with involutive negation. Strictly speaking, we are talking here of *multiplicative-additive* linear logics, i.e., linear logics without *exponentials*. Linear logics in which weakening rule is derivable, are known as *affine* linear logics. These are logics over $\mathbf{FL_{ew}}$. Although linear logic community is rather un-algebraically inclined, we believe that an algebraic analysis of linear logics could be an informative enterprise.

This view may be supported to some extent by the success of algebraic methods in the area of *relevant logics* (cf. e.g., Dunn [21], Restall [49] for more), which are substructural, but they typically have contraction and lack exchange. In most cases they also contain the distributive law, and thus, algebras for relevant logics (sometimes referred to as *de Morgan monoids*) are distributive as lattices.

Fuzzy logics, as we said, fit entirely within our framework, not only as extensions of $\mathbf{FL_{ew}}$, but, and perhaps more importantly, since they can be thought of primarily in algebraic terms. Somewhat paradoxically, this is precisely the reason why we defer their introduction until later.

Now, as a first approximation at a systematic description of the class of logics over $\mathbf{FL_{ew}}$ we will introduce some finitely axiomatisable landmarks. Consider the following axioms, which *for the first and last time* we will explicitly present in their "logical" as well as "algebraic" guises, respectively in the left-middle and right-middle columns. Readers will notice slight changes in form, but will find no difficulty in convincing themselves that they do not change the content (i.e., result in equivalent formulas).

EM:	$\alpha \vee \neg\alpha$	$x \vee \neg x = 1$	(excluded middle),
DN:	$\neg\neg\alpha \to \alpha$	$\neg\neg x = x$	(double negation),
Con:	$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$	$x^2 = x$	(contraction),
WCon:	$(\alpha \to \neg\alpha) \to \neg\alpha$	$\neg x^2 = \neg x$	(weak contraction),
P:	$((\alpha \to \beta) \to \alpha) \to \alpha$	$(x \to y) \to x \leq x$	(Peirce law),
WP:	$(\neg\alpha \to \alpha) \to \alpha$	$\neg x \to x \leq x$	(weak Peirce law),
Lin:	$(\alpha \to \beta) \vee (\beta \to \alpha)$	$(x \to y) \vee (y \to x) = 1$	(linearity),
Dis:	$(\alpha \wedge (\beta \vee \gamma)) \to ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	(distributive law).

Note that WCon and WP are obtained from Con and P, respectively, by replacing β by 0. The axiom Con plays essentially the same role as the contraction rule. It is easy to see that $\mathbf{FL_{ew}}[\mathrm{Con}]$ is equal to the intuitionistic logic $\mathbf{Int}$, which is equal also to $\mathbf{FL_{ew}}$ with the axiom $\alpha \to \alpha^2$. On the other hand, WCon is equivalent to $\neg\alpha^2 \to \neg\alpha$, which is the contrapositive of $\alpha \to \alpha^2$, and moreover $\mathbf{FL_{ew}}[\mathrm{WCon}]$ is equal to $\mathbf{FL_{ew}}$ with the axiom $\neg(\alpha \wedge \neg\alpha)$. The logic $\mathbf{FL_{ew}}[\mathrm{DN}]$ is equal to the logic without contraction investigated by Grišin. Méndez and Salto discussed Kripke-type semantics for (the fusion-free fragment of) $\mathbf{FL_{ew}}[\mathrm{Lin}]$ and $\mathbf{FL_{ew}}[\mathrm{Lin}, \mathrm{WCon}]$ in [41], and for $\mathbf{FL_{ew}}[\mathrm{Dis}, \mathrm{WCon}]$ in [42]. Esteva and Godo [22] discuss quite comprehensively some algebraic aspects of $\mathbf{FL_{ew}}[\mathrm{Lin}]$, which they call *monoidal t-norm based logic* or MTL.

Lemma 1.8. *Each of the logics* $\mathbf{FL_{ew}}[\mathrm{WP}]$, $\mathbf{FL_{ew}}[\mathrm{P}]$, $\mathbf{FL_{ew}}[\mathrm{EM}]$ *and* $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{DN}]$ *is equal to the classical logic* $\mathbf{Cl}$.

Proof. As mentioned above, any instance of WP is provable in $\mathbf{FL_{ew}}$[P]. It is obvious that any instance of P is provable in **Cl**. To show that each formula φ provable in **Cl** is also provable in $\mathbf{FL_{ew}}$[EM], it is enough to show that Con is provable in $\mathbf{FL_{ew}}$[EM]. For, $\mathbf{FL_{ew}}$[Con] is equal to **Int** and **Int**[EM] is equal to **Cl**. We note that the formula $(\neg\gamma \vee \delta) \leftrightarrow (\gamma \to \delta)$ is provable in $\mathbf{FL_{ew}}$[EM], using the fact that

$$(\gamma \vee \neg\delta) \to ((\gamma \to \delta) \to (\neg\gamma \vee \delta))$$

is provable in $\mathbf{FL_{ew}}$. Thus,

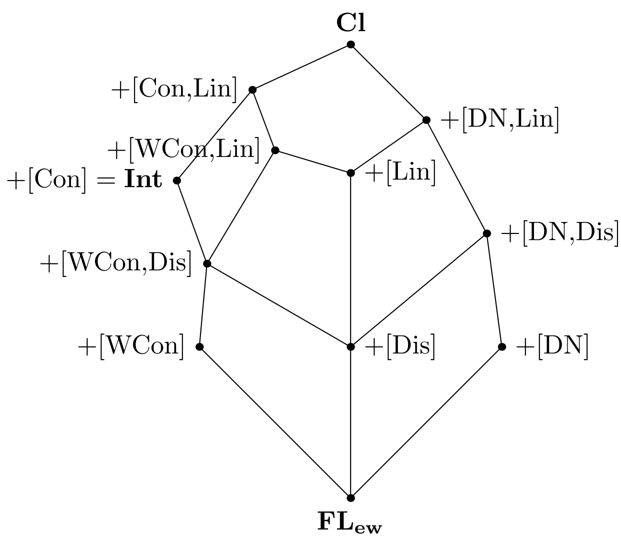
$$((\alpha \to (\alpha \to \beta))) \to (\alpha \to \beta)) \leftrightarrow ((\neg\alpha \vee \neg\alpha \vee \beta) \to (\neg\alpha \vee \beta))$$

is provable. But, the right-hand side of the above equivalence is also provable. Hence, Con is provable in $\mathbf{FL_{ew}}$[EM]. To show that EM is provable in $\mathbf{FL_{ew}}$[WP], let ψ be $\alpha \vee \neg\alpha$. Now, since $\neg\psi \to \psi$ is provable in $\mathbf{FL_{ew}}$ and $(\neg\psi \to \psi) \to \psi$ is an instance of WP, we have that ψ, i.e. $\alpha \vee \neg\alpha$ is provable in $\mathbf{FL_{ew}}$[WP].

Finally, we will show that $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{DN}]$ is equal to **Cl**. Since $\mathbf{FL_{ew}}$[Con] is equal to **Int**, it is enough to show that $\alpha \to \alpha^2$ is provable in $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{DN}]$. By WCon, $\neg\alpha^2 \to \neg\alpha$ is provable. By taking the contraposition and using DN, $\alpha \to \alpha^2$ follows. This completes our proof.

Note that even $\mathbf{FL_{e}}$[P] is equal to **Cl** as the weakening rule is derivable in it (thanks to K. Fujita for pointing this out). A part of the above lemma is essentially obtained in Theorem 7.31 of [55], which says that complemented residuated lattices are exactly same as Boolean algebras. It can be shown (by using e.g. the cut elimination theorem) that Grišin logic $\mathbf{FL_{ew}}$[DN] is strictly weaker than the classical logic, and hence EM is not provable in $\mathbf{FL_{ew}}$[DN]. Also, we can show that EM is provable in $\mathbf{FL_{e}}[\mathrm{WCon}, \mathrm{DN}]$. On the other hand, it can be shown that DN is not provable in $\mathbf{FL_{ec}}$[EM].

The picture below shows the inclusion relationship between the logics mentioned above. In fact, these form an upper subsemilattice of the lattice of logics over $\mathbf{FL_{ew}}$, i.e., all the joins suggested by the picture are "real". The meets however are far from being so, and thus the picture does not show the whole of the sublattice generated by the logics in question. Obtaining the full Hasse diagram of this sublattice might be of some interest (it will be finite, by local finiteness of distributive lattices).



Lemma 1.9. *All inclusions shown in the picture above are proper. Moreover, all joins in the picture coincide with joins in the lattice of logics over* $\mathbf{FL_{ew}}$.

Proof. It is well-known that $\mathbf{FL_{ew}}[\mathrm{Con}, \mathrm{Lin}]$, i.e. $\mathbf{Int}[\mathrm{Lin}]$, is strictly weaker than $\mathbf{Cl}$. Thus, $\mathbf{FL_{ew}}[\mathrm{Lin}]$, $\mathbf{FL_{ew}}[\mathrm{Dis}]$, and $\mathbf{FL_{ew}}$ are strictly weaker than $\mathbf{FL_{ew}}[\mathrm{DN}, \mathrm{Lin}]$, $\mathbf{FL_{ew}}[\mathrm{DN}, \mathrm{Dis}]$ and $\mathbf{FL_{ew}}[\mathrm{DN}]$, respectively.

Also, it is well-known that $\mathbf{Int}[\mathrm{Lin}]$ is strictly stronger than $\mathbf{Int}$. Thus, logics $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{Lin}]$ and $\mathbf{FL_{ew}}[\mathrm{Lin}]$ are strictly stronger than $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{Dis}]$ and $\mathbf{FL_{ew}}[\mathrm{Dis}]$, respectively.

We will show that Lin is not provable in $\mathbf{FL_{ew}}[\mathrm{DN}, \mathrm{Dis}]$. Let $\mathbf{M}$ be the residuated lattice with the universe $\{0, d, c, b, a, 1\}$ such that $0 < d < b < a < 1$, $0 < d < c < a < 1$ but b is incomparable with c. Moreover, assume that $a^2 = a, b^2 = b, c^2 = c, a \cdot b = b, a \cdot c = c, a \cdot d = 0$ (thus, $x \cdot d = 0$ if $x < a$) and $b \cdot c = 0$. Then, $\mathbf{M}$ is an si distributive residuated lattice satisfying $\neg\neg x = x$ for any x. It is not linearly ordered, thus falsifies Lin.

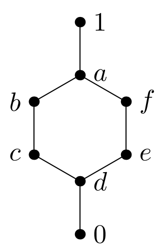
To see that $\mathbf{FL_{ew}}[\mathrm{DN}, \mathrm{Lin}]$ is strictly weaker than $\mathbf{Cl}$, consider the residuated lattice $\mathbf{Ł_3}$ with the universe $\{0, a, 1\}$ such that $0 < a < 1$ and $a^2 = 0$ (it corresponds to the three-valued Łukasiewicz logic, hence the name). Since $\neg a = a$ holds, DN is valid but EM is not.

From this it follows that $\mathbf{FL_{ew}}[\mathrm{Lin}]$ is strictly weaker than $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{Lin}]$. For, if WCon were provable in $\mathbf{FL_{ew}}[\mathrm{Lin}]$ then it would be provable in $\mathbf{FL_{ew}}[\mathrm{DN}, \mathrm{Lin}]$. Then, EM should be derived in $\mathbf{FL_{ew}}[\mathrm{DN}, \mathrm{Lin}]$, since by Lemma 1.8 it is provable in $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{DN}]$. This is a contradiction. Similarly, we can show that $\mathbf{FL_{ew}}[\mathrm{Dis}]$ and $\mathbf{FL_{ew}}$ are strictly weaker than $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{Dis}]$ and $\mathbf{FL_{ew}}[\mathrm{WCon}]$, respectively.

We will show next that Con is not provable in $\mathbf{FL_{ew}}[\mathrm{WCon}, \mathrm{Lin}]$. Let $\mathbf{N}$ be the residuated lattice with the universe $\{0, b, a, 1\}$ such that $0 < b < a < 1$ and $b = a^2 = a^3$. It is easy to see that $a \to b = a$ and $\neg a = \neg b = 0$. It is clear that WCon is valid in a given residuated lattice if and only if $\neg x^2 = \neg x$ for any x in it. So, WCon is valid in $\mathbf{N}$. On the other hand, Con is not valid in it, as $a \to a^2 = a < 1$. As a consequence, Con is not provable in $\mathbf{FL_{ew}}[\mathrm{WCon}]$, either.

To show that Dis is not provable in $\mathbf{FL_{ew}}[\mathrm{DN}]$ consider the residuated lattice $\mathbf{O}$ with the universe $\{0, f, e, d, c, b, a, 1\}$. Lattice ordering is such that $0 < d < c < b < a < 1$ and $0 < d < e < f < a < 1$, and elements from $\{b, c\}$ and $\{e, f\}$ are pairwise incomparable. Multiplication is given by the table below, and the lattice ordering is shown beside it.

	1	a	b	c	d	e	f	0
1	1	a	b	c	d	e	f	0
a	a	c	c	c	0	d	d	0
b	b	c	c	c	0	0	d	0
c	c	c	c	c	0	0	0	0
d	d	0	0	0	0	0	0	0
e	e	d	0	0	0	d	d	0
f	f	d	d	0	0	d	d	0
0	0	0	0	0	0	0	0	0

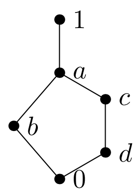


Clearly, $\mathbf{O}$ contains $\{f, e, d, c, b, a\}$ as a sublattice, and that is a copy of the so-called *benzene ring*, which shows that $\mathbf{O}$ is not distributive, and even not modular. But it is easy to see that $\neg a = d$, $\neg b = e$, $\neg c = f$,

$\neg d = a$, $\neg e = b$, and $\neg f = c$; hence DN holds in **O**. It follows that Dis in not provable in $\mathbf{FL_{ew}}$[DN]. Let us note here that in [26], Grišin gave a cut-free system for $\mathbf{FL_{ew}}$[DN]. This system yields an easy alternative proof that Dis is not provable in $\mathbf{FL_{ew}}$[DN].

It remains to show that Dis is not provable in $\mathbf{FL_{ew}}$[WCon]. Let **P** be the residuated lattice with the universe $\{0, d, c, b, a, 1\}$ such that $0 < b < a < 1$, $0 < d < c < a < 1$ and c and d are incomparable with b. As previously, multiplication is given by the table below, and the lattice ordering is shown beside it.

	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	a	b	d	d	0
b	b	b	b	0	0	0
c	c	d	0	d	d	0
d	d	d	0	d	d	0
0	0	0	0	0	0	0



Thus, **P** is obtained from the nonmodular lattice $\mathbf{N_5}$, called the *pentagon*, by adding the new greatest element 1. Then, **P** is an si nondistributive residuated lattice. On the other hand, it is easy to see that $\neg a = 0, \neg b = c$ and $\neg c = \neg d = b$. Using this, we can show that WCon is valid in **P**. Thus, Dis is not provable in $\mathbf{FL_{ew}}$[WCon].

Finally, it follows from Proposition 1.10 that the joins are faithfully shown in the picture.

Worthwhile classes of algebras around

Residuated structures have been of interest to algebraists ever since universal algebra gained recognition as a branch of mathematics (and maybe even before, witness Dedekind's work on lattices of ideals of rings). Of these we will only mention a few, directly related to residuated lattices.

One such is the class of *BCK-algebras* called thus because of the connection with combinators B, C, and K (cf. Curry [17]). These turn out to be precisely $\{\to, 1\}$-subreducts of residuated lattices. Iseki and Tanaka [30] is a classic, although admittedly outdated, introduction to BCK-algebras. Blok and Raftery [10] contains quite a few new results and comprehensive bibliography. Unfortunately, to our best knowledge no up-to-date monograph of the subject exists. One outstanding algebraic feature of BCK-algebras is that they are not a variety (cf. Wroński [57]), but only a (finitely axiomatised) quasivariety.

A finitely axiomatised quasivariety results as well, when we take $\{\cdot, \to, 1\}$-subreducts of residuated lattices. These are better known as *partially ordered, commutative, residuated, integral monoids* or *pocrims*. Again, Blok and Raftery [10], and also [11]) is a good reference. A subclass of pocrims, called *hoops*, defined (relative to the quasivariety of pocrims) by the identity $x(x \to y) = y(y \to x)$, has been comprehensively studied in Blok and Ferreirim [4]. Hoops form a variety, and are semilattice ordered in a natural way, with $x \wedge y$ defined as $x(x \to y)$. If a hoop satisfies also: $x \to (x \to y) = y \to (y \to x)$ then the ordering is actually a lattice ordering, with the join $x \vee y$ defined as $x \to (x \to y)$. Such hoops are called *Wajsberg hoops*, and as we have seen are a subvariety of residuated lattices (precisely speaking, are term-equivalent to such a subvariety). This particular subvariety of residuated lattices has been rediscovered many times under many names. Thus, for instance, *MV-algebras* in Chang[15], *Wajsberg algebras* in Font, Rodrígues and Torrens [24], *CN-algebras* in Komori [33], and *bounded commutative BCK-algebras* in Iseki and Tanaka [30]. Typical examples of Wajsberg algebras are *Łukasiewicz algebras* called thus because of their being algebras of truth values in Łukasiewicz many-valued logics. These are algebras whose universes are certain subsets of the real interval $[0,1]$, and operations defined by:

(i) $x \to y = \min\{1, 1-x+y\}$,

(ii) $\neg x = x \to 0 = 1-x$,

(iii) $x \cdot y = \neg(x \to \neg y) = \max\{0, x+y-1\}$.

The infinite Łukasiewicz algebra $\mathbf{Ł}$ has the whole interval $[0,1]$ as its universe, whereas the finite algebra $\mathbf{Ł}_{n+1}$, for a positive integer n, is defined on the universe $\{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$.

Now, we have come to a convenient place to encounter fuzzy logic. The underlying idea of fuzziness is that the the truth-value of a proposition can be any number from $[0,1]$. Certain considerations about how such fuzzy truth-values should relate to one another in order to preserve some basic intuitive properties lead to the notion of a *triangular norm* (t-norm), borrowed from the theory of probabilistic measures. A t-norm is a binary function T from $[0,1] \times [0,1]$ to $[0,1]$, and satisfying:

(i) $T(x, T(y,z)) = T(T(x,y), z)$,

(ii) $T(x,y) = T(y,x)$,

(iii) $T(x,1) = x$,

(iv) $x \leq y$ implies $T(x,z) \leq T(y,z)$.

Thus, setting $x \cdot y = T(x,y)$, we get a commutative monoid $\langle [0,1], \cdot, 1\rangle$, with multiplication monotone with respect to the natural order. Further, T is called *continuous t-norm* iff it is continuous as a function. There are three typical examples of continuous t-norms:

(1) *Lukasiewicz* t-norm: $T(x,y) = \max\{0, x+y-1\}$,

(2) *Gödel* t-norm: $T(x,y) = \min\{x,y\}$,

(3) *product* t-norm: $T(x,y) = xy$, (xy being ordinary multiplication of reals);

and it turns out that in an important sense they are the only ones. Roughly speaking, each continuous t-norm behaves locally as one of the three.

Now, for any $y, z \in [0,1]$, define a subset $I_{y,z}$ of [0,1] by $I_{y,z} = \{x : T(x,y) \leq z\}$ and let $u = \sup I_{y,z}$. By continuity of T, we have that $T(u,y) = \sup\{T(x,y) : x \in I_{y,z}\} \leq z$. Thus, u is in fact equal to $\max I_{y,z}$, and from this it follows that

$$T(x,y) \leq z \text{ iff } x \leq \max I_{y,z}.$$

Putting, $y \to z = \max I_{y,z}$, we obtain a residuated lattice $\big\langle [0,1]; \wedge, \vee, \cdot, \to, 0, 1\big\rangle$, with meet and join defined in the natural way as minimum and maximum (cf. remarks after Proposition 1.2). It turns out that algebras of this form suffice as an adequate semantics for fuzzy logics. In particular, the variety generated by all such algebras forms a natural algebraic counterpart for Hajek's *basic logic* (**BL**, cf. [27]), that has long been considered the smallest fuzzy logic. Because of this connection, the algebras from this variety are sometimes called *basic algebras* in the fuzzy logic community.

Note however, that only the *left-continuity* of T, i.e., $T(\sup_i x_i, y) = \sup_i T(x_i, y)$, as opposed to *right-continuity*: $T(\inf_i x_i, y) = \inf_i T(x_i, y)$, is used for showing the existence of residuation. Now, on the one hand, it is easily seen that when $\cdot$ is defined by $x \cdot y = T(x, y)$ for all x, y with a left-continuous t-norm T, the structure $\langle [0, 1], \cdot, 1\rangle$ is nothing but an *integral, unital commutative quantale* with the underlying set $[0, 1]$ (see [50] for more on quantales; the name is not misleading, they did arise out of some considerations around quantum theory). On the other hand, Esteva and Godo [22] have recently investigated the variety generated by such algebras as a natural candidate for a minimal fuzzy logic weaker than Hajek's (they call this logic **MTL**, cf. above). Indeed, if follows by remarks after Proposition 1.2 that left-continuity is both necessary and sufficient for a t-norm to have a (unique) residual. It turns out that the variety of MTL-algebras is precisely the variety of linearly ordered residuated lattices.

One may wonder what varieties are generated by algebras made up of the three continuous t-norms above. Here is the answer. Łukasiewicz t-norm generates the variety of Wajsberg algebras, Gödel t-norm generates the variety of linearly ordered Heyting algebras, and product t-norm generates the variety of so-called *product algebras*, little known outside the fuzzy logic community, but quite interesting nevertheless. For instance, it is shown in [16] that this variety covers the variety of Boolean algebras, i.e., its lattice of subvarieties is a three element chain. It follows that all finite product algebras are Boolean algebras, and thus the logic associated with this variety (called, surely enough, *product logic*) has a kind of anti-finite model property. Namely, all classical tautologies which are not theorems of product logic, have only infinite countermodels.

Since the variety $\mathcal{R}$ is big and nasty (there will be some justification for this statement later on), we want some means of classifying its subvarieties. We believe the following proposal to be quite natural.

For each $n \in \omega$, put:

$$x^n = x^{n+1}, \tag{E$_n$}$$

and

$$x \vee \neg x^n = 1. \tag{EM$_n$}$$

For notational convenience, we adopt the convention that $x^0 = 1$. Notice, that (E_n) is equivalent over residuated lattices to:

$$\underbrace{x \to (x \to \ldots (x}_{n \text{ times}} \to y) \ldots) = \underbrace{x \to (x \to \ldots (x}_{n+1 \text{ times}} \to y) \ldots),$$

which from the point of view of logics represents a weak form of contraction. For $n = 1$, we get the usual contraction axiom, equivalent to the contraction rule. Analogously, (EM_n) is a weak form of excluded middle; again, for $n = 1$, we get the classical logical law of excluded middle. Let $\mathcal{E}_n$ and $\mathcal{EM}_n$ stand for subvarieties of $\mathcal{R}$ defined respectively by (E_n) and (EM_n). Clearly, (E_0) and (EM_0) both define the trivial variety. It is immediate, that $\mathcal{E}_n \subseteq \mathcal{E}_{n+1}$, $\mathcal{EM}_n \subseteq \mathcal{EM}_{n+1}$, and $\mathcal{EM}_n \subseteq \mathcal{E}_n$. It is also not difficult to find examples showing that all these inclusions are proper (except the last one for $n = 0$). Let us also note that all the varieties $\mathcal{EM}_n$ are semisimple. To see that take an si algebra $\mathbf{A}$ from $\mathcal{EM}_n$, and any $b \in A - \{1\}$. By (EM_n) we have $b \vee \neg b^n = 1$. As $b < 1$, Proposition 1.4 yields that $\neg b^n = 1$, hence $b^n = 0$. Then the claim follows by Lemma 1.1. Incidentally, it also shows that $\mathcal{EM}_n \subseteq \mathcal{E}_n$.

Observe also that $\mathcal{EM}_1$ is the variety of Boolean algebras, and $\mathcal{E}_1$ is the variety of Heyting algebras (see the proof of Lemma 1.8). Later on we will show that both the varietal joins $\bigvee_{n\in\omega} \mathcal{E}_n$ and $\bigvee_{n\in\omega} \mathcal{EM}_n$ are equal to the whole $\mathcal{R}$.

Of other natural subvarieties of $\mathcal{R}$, three will be so often mentioned later on that they deserve short names of their own. These are the varieties of *distributive*, *linear*, and *classical* residuated lattices, defined relative to $\mathcal{R}$ respectively by the identities:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{D}$$

$$(x \to y) \vee (y \to x) = 1, \tag{L}$$

$$x = \neg\neg x. \tag{C}$$

We will call them: $\mathcal{D}$, $\mathcal{L}$, and $\mathcal{C}$.

We finish off with two easy observations about varieties $\mathcal{E}_n$ and $\mathcal{EM}_n$.

Proposition 1.11. *Let $\mathcal{V}$ be a variety of residuated lattices. Then $\mathcal{V}$ has EDPC iff $\mathcal{V} \subseteq \mathcal{E}_n$, for some $n \in \omega$.*

Proof. It is easy to see that $(c, d) \in \mathbf{Cg}(a, b)$ iff for some $k \in \omega$ we have $((a \to b)(b \to a))^k \leq (c \to d)(d \to c)$ iff $(c \to d)(d \to c)$ belongs to the congruence filter generated by $(a \to b)(b \to a)$. Suppose $\mathcal{V}$ verifies $x^{n+1} = x^n$ for an $n \in \omega$. Then, $((a \to b)(b \to a))^{n+1} = ((a \to b)(b \to a))^n$ and thus, $(c, d) \in \mathbf{Cg}(a, b)$ iff $((a \to b)(b \to a))^n \leq (c \to d)(d \to c)$, which can obviously be rendered as an equation and thus EDPC follows.

For the converse, suppose $\mathcal{V}$ has EDPC, yet falsifies $x^n = x^{n+1}$ for all $n \in \omega$. Consider the sequence $\langle \mathbf{A}_n : n \in \omega\rangle$ of algebras from $\mathcal{V}$ such that $\mathbf{A}_n$ falsifies $x^{n+1} = x^n$ and let $a_n \in A_n$ be an element witnessing this fact. Now, by EDPC, there is a a first-order conjunction of identities $\phi(x, y, z, t)$ such that for all $a, b, c, d \in \mathbf{A} \in \mathcal{V}$ we have $(c, d) \in \mathbf{Cg}(a, b)$ iff $\mathbf{A} \models \phi(a, b, c, d)$. In particular, for any $n \in \omega$, $\mathbf{A}_n \models \phi(a_n, 1, a_n^{n+1}, 1)$, as this simply says that a_n^{n+1} belongs to the filter generated by a_n which is always the case. Now, as $\phi(x, y, z, t)$ is a universal Horn formula, it carries over to direct products. Consider the product $\mathbf{B} = \prod_{n\in\omega} \mathbf{A}_n$. Let $a = \langle a_n : n \in \omega\rangle$, and $b = \langle a_n^{n+1} : n \in \omega\rangle$. Since $\phi(a_n, 1, a_n^{n+1}, 1)$ holds on all coordinates, $\mathbf{B} \models \phi(a, 1, b, 1)$. Thus, by EDPC, $(b, 1) \in \mathbf{Cg}(a, 1)$, which means that b belongs to the congruence filter generated by a. This, in turn, means that, for some $k \in \omega$, $\mathbf{B} \models a^k \leq b$. Let $u(i)$ stand for the projection of u on the i-th coordinate. On the $k+1$-th coordinate we have: $a^k(k+1) = a_{k+1}^k > a_{k+1}^{k+1} = b(k+1)$. This yields a contradiction, which ends the proof.

Proposition 1.12. *Let $\mathcal{V}$ be a variety of residuated lattices. Then, $\mathcal{V}$ is a discriminator variety, if $\mathcal{V} \subseteq \mathcal{EM}_n$, for some $n \in \omega$.*

Proof. Since $\mathcal{V} \models x \vee \neg x^n = 1$, for some $n \in \omega$, we have that for any si algebra $\mathbf{A} \in \mathcal{V}$ and any element $a \in A$ either $a = 1$ or else $a^n = 0$. Consider the term: $t(x, y, z) = ((x \leftrightarrow y)^n \wedge z) \vee (\neg(x \leftrightarrow y)^n \wedge x)$. It is easily checked that t is a discriminator term.

Question 1.1. *Is the converse of Proposition 1.12 true?*

2. GENERATION BY FINITE SIMPLE ALGEBRAS

In this chapter, we will show that the variety $\mathcal{R}$ of residuated lattices is generated by its finite simple members. This improves upon a result of Okada and Terui [44] where generation by finite members (better known to logicians as *finite model property*) is shown. Our proof combines a proof theoretic argument with an algebraic one. The proof theoretic part, which shows generation by simple algebras, is modelled after an argument of Grišin [26]. The algebraic part, which proves that simple residuated lattices have the *finite embeddability property*, employs a filtration-like reasoning that starts off from a locally finite reduct of a (special) residuated lattice and builds up a finite residuated lattice on the universe of a finitely generated subreduct. The whole chapter is based on our [36].

A characterisation of semisimplicity

An algebra $\mathbf{A}$ is *semisimple* if $\mathbf{A}$ has a subdirect representation with simple factors. Let Φ be the set of all maximal filters of a residuated lattice $\mathbf{A}$. Recall that a filter F is called maximal iff F is maximal among proper filters of $\mathbf{A}$. It is easy to see that a filter F is maximal iff for each $u \in A$ either $u \in F$ or there is a $k \geq 1$ such that $\neg u^k \in F$. Define the *radical* Rad_A of $\mathbf{A}$ by $\mathrm{Rad}_A = \bigcap_{F\in\Phi} F$. Then, since maximal filters correspond to maximal congruences, it follows immediately that a residuated lattice $\mathbf{A}$ is semisimple if and only if $\mathrm{Rad}_A = \{1\}$.

For elements $a, b \in A$, set $a+b$ to be $\neg(\neg a \cdot \neg b)$. Note that this operation, while commutative, is usually not associative. Moreover, while $a + a = \neg(\neg a \cdot \neg a) = \neg(\neg a)^2$ holds, neither $a + (a + a)$, nor $(a + a) + a$ yields the desired $\neg(\neg a)^3$. To remedy that, for any positive integer m we define $\widetilde{m}a$ to be $\neg(\neg a)^m$. Note that $\widetilde{1}a = \neg\neg a$. (On the margin: it may be interesting to have a closer look at residuated lattices for which $+$ *is* associative; for example, all classical residuated lattices, all Heyting algebras, and all product algebras are such.)

Lemma 2.1. *For any x in a given residuated lattice $\mathbf{A}$, $x \in \mathrm{Rad}_A$ if and only if for any $n \geq 1$ there exists $m \geq 1$ such that $\widetilde{m}(x^n) = 1$.*

Proof. Assume that for any $n \geq 1$ there exists $m \geq 1$ such that $\widetilde{m}(x^n) = 1$. Suppose $x \notin \mathrm{Rad}_A$. Thus, there is a maximal filter F with $x \notin F$. Since F is maximal, there is a $k \geq 1$ such that $\neg x^k \in F$. Further, by the assumption, there is an $l \geq 1$ for which $\widetilde{l}(\neg x^k) = 1$, i.e., $(\neg x^k)^l = 0$. This implies $0 \in F$, which contradicts the fact that F is proper.

Conversely, suppose that there exists $n \geq 1$ such that $\widetilde{m}(x^n) \neq 1$ for any m. If $(\neg(x^n))^m = 0$ then $\widetilde{m}(x^n) = \neg(\neg(x^n))^m = 1$, which is a contradiction. Thus, $(\neg(x^n))^m > 0$ for any m. Let $z = \neg(x^n)$ and H be the filter generated by z. Clearly, H is proper as $z^m > 0$ for any m. By Zorn lemma, there exists a maximal filter G such that $H \subseteq G$. Now, suppose that $x \in G$. Then x^n must be also in G. But this is a contradiction, since $z = \neg(x^n) \in G$. Hence, $x \notin G$ and therefore it doesn't belong to Rad_A.

Corollary 2.1. *A residuated lattice* $\mathbf{A}$ *is semisimple iff for every* $a \in A \setminus \{1\}$ *there is an* $N \geq 1$ *such that for any* $m \geq 1$, *we have:* $\widetilde{m}a^N < 1$.

Sequent calculi for $\mathbf{FL_{ew}}$

Before we put Corollary 2.1 to better use in the context of free residuated lattices, we prepare tools for reasoning about these. Since free residuated lattices are precisely Lindenbaum algebras of the logic $\mathbf{FL_{ew}}$, our tools will have the shape of two sequent calculi for this logic. First, we will introduce a calculus $\mathbf{SFL_{ew}}$, which is essentially the same as the one introduced in the previous chapter, but with some convenient modifications. We will slightly change the form of initial sequents and generalise the form of rules for fusion. This will allow us to drop the weakening rule. Further, we will view sequents as *multisets* of formulas, rather than *lists* of them, which will let us drop the exchange rule as well, so that no structural rules get in the way. In our new rules for fusion, we will allow formulas of the form $\alpha_1 \cdot \ldots \cdot \alpha_m$ for any $m \geq 2$, with no conventions about implicit bracketing. This is harmless, since the associativity of fusion is provable in $\mathbf{FL_{ew}}$.

Now, a *sequent* of $\mathbf{SFL_{ew}}$ is of the form $\Gamma \Rightarrow \beta$, where Γ is a possibly empty multiset of formulas. Note that the right-hand side of $\Rightarrow$ must always exist. As usual, when Γ is a multiset $\{\alpha_1, \ldots, \alpha_m\}$, the sequent $\Gamma \Rightarrow \beta$ is also expressed as $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$. Also, the multiset union of multisets Γ and Δ (and of $\{\alpha\}$ and Δ) is denoted by Γ, Δ (and α, Δ, respectively). In what follows, both Γ and Δ denote arbitrary multisets of formulas. The system $\mathbf{SFL_{ew}}$ consists of the following initial sequents:

1. $x, \Gamma \Rightarrow x$ for any variable x
2. $0, \Gamma \Rightarrow \gamma$

and the following rules of inference:

Cut rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma}$$

Rules for logical connectives:

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \to \beta, \Gamma, \Delta \Rightarrow \gamma}\ (\to\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 2)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee \beta, \Gamma \Rightarrow \gamma}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma}\ (\wedge 1 \Rightarrow) \qquad \frac{\beta, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma_1 \Rightarrow \alpha_1 \quad \ldots \quad \Gamma_m \Rightarrow \alpha_m}{\Gamma_1, \ldots, \Gamma_m \Rightarrow \alpha_1 \cdot \ldots \cdot \alpha_m}\ (\Rightarrow \cdot) \qquad \frac{\alpha_1, \ldots, \alpha_m, \Gamma \Rightarrow \gamma}{\alpha_1 \cdot \ldots \cdot \alpha_m, \Gamma \Rightarrow \gamma}\ (\cdot \Rightarrow)$$

The formula introduced in a rule I, is called the *principal* formula of I; formulas in upper sequent(s) of I which become components of the principal formula are called *auxiliary* formulas. Other formulas, i.e., the formula γ and formulas in Γ, Γ_i, Δ, are called *side* formulas.

As usual, we say that a formula α is provable in $\mathbf{SFL_{ew}}$ if the sequent $\Rightarrow \alpha$ is provable in it. Since the cut elimination theorem holds for the calculus $\mathbf{SFL_{ew}}$ (the proof from[43] works, with obvious modifications), the cut rule is redundant. Next, we will introduce another sequent calculus $\mathbf{SFL^+_{ew}}$, which will then be shown to be equivalent to $\mathbf{SFL_{ew}}$. We will say that α is a *fusion-formula* if the outermost connective of α is fusion. Now, $\mathbf{SFL^+_{ew}}$ is the system obtained from $\mathbf{SFL_{ew}}$ by deleting the cut rule and by adding the following condition (+) in any application of the rules $(\Rightarrow \cdot)$ and $(\cdot \Rightarrow)$:

$$\textit{none of } \alpha_i s \textit{ are fusion-formulas.} \qquad (+)$$

For double precision we note here that when we set out to show the equivalence of $\mathbf{SFL^+_{ew}}$ to $\mathbf{SFL_{ew}}$, we will identify each formula of the form $\alpha_1 \cdot \ldots \cdot \alpha_m$ of $\mathbf{SFL^+_{ew}}$ with an arbitrary formula of $\mathbf{SFL_{ew}}$ obtained from $\alpha_1 \cdot \ldots \cdot \alpha_m$ by introducing any proper bracketing in it. As we have already said, this does not cause problems since all such formulas are equivalent to one another over $\mathbf{FL_{ew}}$.

Lemma 2.2. *A sequent $\Gamma \Rightarrow \delta$ is provable in $\mathbf{SFL_{ew}}$ if and only if it is provable in $\mathbf{SFL^+_{ew}}$.*

Proof. It is enough to show that if a sequent $\Gamma \Rightarrow \delta$ is provable in $\mathbf{SFL_{ew}}$ then it is provable in $\mathbf{SFL^+_{ew}}$. In other words, it suffices to show that any application of $(\Rightarrow \cdot)$ and $(\cdot \Rightarrow)$ not satisfying the condition (+) in a given cut-free proof P of $\Gamma \Rightarrow \delta$ can be replaced by one with (+). Let β be any formula of the form $\beta_1 \cdot \ldots \cdot \beta_k$ $(k > 0)$, where none of β_js are fusion-formulas. Then we define the *degree* $d(\beta)$ of β by $d(\beta) = k - 1$, i.e. the number of outermost occurrences of fusion in β. Next, when I is either an application of $(\cdot \Rightarrow)$ of the form

$$\frac{\alpha_1, \ldots, \alpha_m, \Gamma \Rightarrow \gamma}{\alpha_1 \cdot \ldots \cdot \alpha_m, \Gamma \Rightarrow \gamma}$$

or an application of $(\Rightarrow \cdot)$ of the form

$$\frac{\Gamma_1 \Rightarrow \alpha_1 \quad \ldots \quad \Gamma_m \Rightarrow \alpha_m}{\Gamma_1, \ldots, \Gamma_m \Rightarrow \alpha_1 \cdot \ldots \cdot \alpha_m}$$

we define the *degree* $d(I)$ of I by $d(I) = d(\alpha_1) + \ldots + d(\alpha_m)$. It is obvious that an application I of either $(\cdot \Rightarrow)$ or $(\Rightarrow \cdot)$ satisfies the condition $(+)$ if and only if $d(I) = 0$.

Assume first that there exists an application of $(\cdot \Rightarrow)$ not satisfying the condition $(+)$ in P. Let us take one of the uppermost applications among them, which we call J. Obviously, $d(J) > 0$. We suppose that the lower sequent of J is $\alpha_1 \cdot \ldots \cdot \alpha_m, \Gamma \Rightarrow \gamma$ with the principal formula $\alpha_1 \cdot \ldots \cdot \alpha_m$. We will show by induction on the degree $d(J)$ that $\alpha_1 \cdot \ldots \cdot \alpha_m, \Gamma \Rightarrow \gamma$ has a cut-free proof in which every application of $(\cdot \Rightarrow)$ satisfies the condition $(+)$.

By our assumption, the degree of one of α_is must be nonzero. Without loss of generality, we can assume that α_1 is of the form $\delta_1 \cdot \ldots \cdot \delta_s$ for $s > 1$. Let Q be the proof of $\alpha_1 \cdot \ldots \cdot \alpha_m, \Gamma \Rightarrow \gamma$ which is a subproof of P. We will trace back ancestors of the auxiliary formula α_1 of J in all branches of Q. Then, we can see that in every sequent in these branches, α_1 is introduced either as the principal formula of an application of $(\cdot \Rightarrow)$ (with $(+)$), or as a side formula of an initial sequent. Now, we replace any ancestor α_1 by the multiset $\delta_1, \ldots, \delta_s$. If one of such α_1 is introduced by an application of $(\cdot \Rightarrow)$ then its lower sequent becomes identical with the upper sequent by this replacement. In such a case, we eliminate this $(\cdot \Rightarrow)$. On the other hand, if it is introduced as a side formula of an initial sequent, the sequent obtained by this replacement remains still an initial sequent. Hence, the figure Q' obtained from Q by this replacement remains a correct proof of $\alpha_1 \cdot \ldots \cdot \alpha_m, \Gamma \Rightarrow \gamma$ whose last inference is an application J' of $(\cdot \Rightarrow)$, with the upper sequent $\delta_1, \ldots, \delta_s, \alpha_2, \ldots, \alpha_n, \Gamma \Rightarrow \gamma$. Since $d(J') = d(J) - d(\alpha_1) < d(J)$, by the hypothesis of induction, $\alpha_1 \cdot \ldots \cdot \alpha_n, \Gamma \Rightarrow \gamma$ has a cut-free proof in which every application of $(\cdot \Rightarrow)$ satisfies $(+)$. In this way, we have a cut-free proof P' of $\Gamma \Rightarrow \delta$ where every application of $(\cdot \Rightarrow)$ satisfies the condition $(+)$.

For example, consider the following (sub)proof whose last inference is an application of $(\cdot \Rightarrow)$ without the condition $(+)$.

$$\dfrac{\dfrac{x \cdot y, z, z \to (x \to t) \Rightarrow z \quad \dfrac{x, y, z, z \to (x \to t) \Rightarrow t}{x \cdot y, z, z \to (x \to t) \Rightarrow t}}{x \cdot y, z, z \to (x \to t) \Rightarrow z \wedge t}}{x \cdot y \cdot z, z \to (x \to t) \Rightarrow z \wedge t}$$

The replacement mentioned above will change the above proof into the following one.

$$\dfrac{\dfrac{x, y, z, z \to (x \to t) \Rightarrow z \quad x, y, z, z \to (x \to t) \Rightarrow t}{x, y, z, z \to (x \to t) \Rightarrow z \wedge t}}{x \cdot y \cdot z, z \to (x \to t) \Rightarrow z \wedge t}$$

Next, we will remove any application of $(\Rightarrow \cdot)$ not satisfying the condition $(+)$ in P'. Suppose that there exists such an application. Similarly to the above, take one of the uppermost applications of $(\Rightarrow \cdot)$ not satisfying $(+)$, called J, which is of the form given above. We will show by induction on $d(J)$ that the lower sequent $\Gamma_1, \ldots, \Gamma_m \Rightarrow \alpha_1 \cdot \ldots \cdot \alpha_m$ has a cut-free proof in which every application of $(\Rightarrow \cdot)$ satisfies $(+)$. Without loss of generality, we can assume that $d(\alpha_1) = k > 0$ and that α_1 is of the form $\delta_1 \cdot \ldots \cdot \delta_k$, where none of δ_js are fusion-formulas. Let R be the proof of $\Gamma_1 \Rightarrow \alpha_1$, which is a subproof of P'. We will trace back the branches in R which consist of sequents having α_1 in the conclusion (such an α_1 must clearly be an ancestor of the α_1 from $\Gamma_1 \Rightarrow \alpha_1$) to the places where this α_1 is introduced. There are two possibilities. It is introduced either by means of an initial sequent of the form: $0, \Delta \Rightarrow \alpha_1$, or as the principal formula of an application of $(\Rightarrow \cdot)$ of the form:

$$\frac{\Delta_1 \Rightarrow \delta_1 \quad \ldots \quad \Delta_k \Rightarrow \delta_k}{\Delta_1, \ldots, \Delta_k \Rightarrow \alpha_1}$$

We will modify the proof R as follows. If α_1 is introduced as the succedent of an initial sequent, we replace this sequent by $0, \Delta, \Gamma_2, \ldots, \Gamma_m \Rightarrow \alpha_1 \cdot \ldots \cdot \alpha_m$, which is still an initial sequent. In the other case, we replace it by:

$$\frac{\Delta_1 \Rightarrow \delta_1 \quad \ldots \quad \Delta_k \Rightarrow \delta_k \quad \Gamma_2 \Rightarrow \alpha_2 \quad \ldots \quad \Gamma_m \Rightarrow \alpha_m}{\Delta_1, \ldots, \Delta_k, \Gamma_2, \ldots, \Gamma_m \Rightarrow \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_m}$$

and put the subproof of each $\Gamma_i \Rightarrow \alpha_i$ in R over it for each $i = 2, \ldots, m$. Note that the degree of the above application of $(\Rightarrow \cdot)$ is $d(J) - d(\alpha_1)$, which is smaller than $d(J)$. Therefore, by the hypothesis of induction the lower sequent of this inference has a cut-free proof in which every application of $(\Rightarrow \cdot)$ satisfies $(+)$. Finally, we replace every sequent $\Sigma \Rightarrow \alpha_1$ in a branch which we have traced, by the sequent $\Sigma, \Gamma_2, \ldots, \Gamma_m \Rightarrow \alpha_1 \cdot \ldots \cdot \alpha_m$. Then, after this replacement, we get a proof R' of $\Gamma_1, \Gamma_2, \ldots, \Gamma_m \Rightarrow \alpha_1 \cdot \ldots \cdot \alpha_m$ in which every application of $(\Rightarrow \cdot)$ satisfies $(+)$. (Note that when $(\to\Rightarrow)$ is used somewhere in these branches, this replacement will be done only for the right upper sequent. Thus, this replacement never cause unnecessary duplications of $\Gamma_2, \ldots, \Gamma_m$.) By repeating this, we have a proof of $\Gamma \Rightarrow \delta$ in $\mathbf{SFL}^+_{\mathbf{ew}}$.

Semisimplicity of free residuated lattices

We will use the characterisation of semisimple residuated lattices from Corollary 2.1. For each formula α, let $l(\alpha)$ denote the length of α (as a sequence of symbols). For a sequence Γ of formulas $\alpha_1, \ldots, \alpha_m$, the length $l(\Gamma)$ of Γ is defined by $l(\Gamma) = l(\alpha_1) + \ldots + l(\alpha_m)$. Also, $\{\neg\alpha^N\}^m$ will stand for the multiset $\neg\alpha^N, \ldots, \neg\alpha^N$ with m times $\neg\alpha^N$.

Lemma 2.3. *Suppose that a formula α is not provable in $\mathbf{SFL}^+_{\mathbf{ew}}$ and that N is any positive integer greater than $l(\alpha)$. Then, for any sequent $\Gamma \Rightarrow \gamma$ such that $l(\Gamma, \gamma) \leq l(\alpha)$ and any positive integer m, if $\{\neg\alpha^N\}^m, \Gamma \Rightarrow \gamma$ is provable in $\mathbf{SFL}^+_{\mathbf{ew}}$ then $\Gamma \Rightarrow \gamma$ is provable in $\mathbf{SFL}^+_{\mathbf{ew}}$.*

Proof. The proof will proceed by double induction on $(m, l(\Gamma, \gamma))$. So, we assume that the lemma holds for $m' < m$ and it also holds for $(m, l(\Delta, \delta))$, whenever $l(\Delta, \delta) < l(\Gamma, \gamma)$. We suppose that α is not provable but $\{\neg\alpha^N\}^m, \Gamma \Rightarrow \gamma$ is provable in $\mathbf{SFL}^+_{\mathbf{ew}}$. Suppose first that $\{\neg\alpha^N\}^m, \Gamma \Rightarrow \gamma$ is an initial sequent. Then, either γ is a propositional variable which occurs also in Γ, or 0 occurs in Γ. It is clear that $\Gamma \Rightarrow \gamma$ is provable in either case.

Next, suppose that the sequent $\{\neg\alpha^N\}^m, \Gamma \Rightarrow \gamma$ is the lower sequent of an inference rule I. We first assume that the principal formula of I is either in Γ or in γ. Then, (each of) upper sequent(s) of I is of the form $\{\neg\alpha^N\}^{m_i}, \Delta_i \Rightarrow \delta_i$ with $m_i \leq m$ and $l(\Delta_i, \delta_i) < l(\Gamma, \gamma)$ (by the *subformula property* and the fact that $\mathbf{SFL^+_{ew}}$ has no contraction rule). Thus, by the hypothesis of induction, (each) $\Delta_i \Rightarrow \delta_i$ is provable. Then $\Gamma \Rightarrow \gamma$ is also provable by applying the same inference rule I.

Finally suppose that the principal formula of I is one of $\neg\alpha^N$. Then, I must be an application of $(\rightarrow\Rightarrow)$. Recall here that a formula $\neg\beta$ is the abbreviation of $\beta \rightarrow 0$. The upper sequents must be of the form $\{\neg\alpha^N\}^{m_1}, \Gamma_1 \Rightarrow \alpha^N$ and $0, \{\neg\alpha^N\}^{m_2}, \Gamma_2 \Rightarrow \gamma$ such that $m_1 + m_2 = m - 1$ and Γ_1, Γ_2 is equal to Γ. Now consider the proof R of the left upper sequent $\{\neg\alpha^N\}^{m_1}, \Gamma_1 \Rightarrow \alpha^N$. As we did in the proof of Lemma 2.2, we trace back branches in R, which consists of sequents having α^N in the conclusion, to the places where these α^N are introduced. It is easy to see that each α^N is introduced either as an initial sequent of the form $0, \Delta \Rightarrow \alpha^N$, or by an application of $(\Rightarrow \cdot)$. Suppose that in at least one place, α^N is introduced by an application J of $(\Rightarrow \cdot)$, whose lower sequent is of the form $\{\neg\alpha^N\}^k, \Sigma \Rightarrow \alpha^N$. Clearly, $k \leq m_1$. We assume here that α is of the form $\delta_1 \cdot \ldots \cdot \delta_w$ such that none of δ_j are fusion-formulas. (Only for the simplicity's sake, we assume in the following that $\delta_1, \ldots, \delta_w$ are mutually distinct.) Then, I must have $N \cdot w$ upper sequents, each of which is of the form $\{\neg\alpha^N\}^{t_i}, \Xi_i \Rightarrow \delta_{n_i}$, where $1 \leq n_i \leq w$, $k = t_1 + \ldots + t_{N \cdot w}$ and the multiset $\Xi_1, \ldots, \Xi_{N \cdot w}$ is equal to Σ. For each j such that $1 \leq j \leq w$, there exist exactly N sequents with the conclusion δ_j among these sequents. We enumerate them as $S^j{}_1, \ldots, S^j{}_N$. Next, for each h such that $1 \leq h \leq N$, take $S^1{}_h, \ldots, S^w{}_h$ for upper sequents and apply $(\Rightarrow \cdot)$ to them. Then, we can get a sequent of the form $\{\neg\alpha^N\}^{u_h}, \Pi_h \Rightarrow \alpha$ for $1 \leq h \leq N$ such that $k = u_1 + \ldots + u_N$ and the multiset $\Pi_1, \ldots, \Pi_N$ is equal to Σ. Now, $l(\Sigma) \leq l(\Gamma_1) \leq l(\Gamma, \gamma) \leq l(\alpha) < N$. If $l(\Pi_h) > 0$ for any h such that $1 \leq h \leq N$ then $l(\Sigma) \geq N$, which is a contradiction. Thus, Π_h must be empty for some h. Let it be f. Then, $\{\neg\alpha^N\}^{u_f} \Rightarrow \alpha$ is provable. By our assumption that α is not provable, u_f must be positive. Then, since $u_f \leq m_1 \leq m - 1 < m$ and $l(\alpha) \leq l(\alpha)$, $\Rightarrow \alpha$ must be provable by the hypothesis of induction. This is a contradiction.

Thus, we have shown that in any place α^N is introduced as an initial sequent of the form $0, \Delta \Rightarrow \alpha^N$. We will modify the proof R of $\{\neg\alpha^N\}^{m_1}, \Gamma_1 \Rightarrow \alpha^N$ as follows. We replace every sequent $\Lambda \Rightarrow \alpha^N$ in a branch which we have traced in R, including initial sequents of the form $0, \Delta \Rightarrow \alpha^N$ mentioned above, by the sequent $\Lambda, \Gamma_2 \Rightarrow \gamma$. Then we will get a proof R^* whose end sequent is $\{\neg\alpha^N\}^{m_1}, \Gamma \Rightarrow \gamma$. Note that $m_1 \leq m-1 < m$. Hence, by the hypothesis of induction, $\Gamma \Rightarrow \gamma$ is provable. This completes the proof.

Lemma 2.4. *If a formula α is not provable in $\mathbf{FL_{ew}}$, there exists a number $N(\geq 1)$ such that $\widetilde{m}(\alpha^N)$ is not provable in $\mathbf{FL_{ew}}$ for any $m \geq 1$.*

Proof. Follows immediately from Lemma 2.3 by taking the sequent $\Rightarrow 0$ for $\Gamma \Rightarrow \gamma$, since $\Rightarrow 0$ is not provable in $\mathbf{SFL^+_{ew}}$.

Theorem 2.1. *Free residuated lattices are semisimple.*

Proof. Remind that free residuated lattices are Lindenbaum algebras of the logic $\mathbf{FL_{ew}}$, and apply Lemma 2.4 to Corollary 2.1.

Corollary 2.2. *The variety of residuated lattices is generated by its simple members.*

Finite embeddability

In this section we show that the class $\mathcal{R}_S$ of simple residuated lattices enjoys the finite embeddability property, which we are now going to define (cf. e.g., [23], for more details).

Let $\mathcal{K}$ be a class of algebras and $\mathbf{P}$ be a partial subalgebra of an algebra $\mathbf{B}$ from $\mathcal{K}$. We say that $\mathbf{B}$ has the *finite embeddability property* in $\mathcal{K}$ iff any finite partial subalgebra of $\mathbf{B}$ can be embedded into a finite algebra $\mathbf{A}$ from $\mathcal{K}$. The class $\mathcal{K}$ has the *finite embeddability property* iff every algebra $\mathbf{B}$ from $\mathcal{K}$ has the finite embeddability property in $\mathcal{K}$.

Thus, our task is to produce for any $\mathbf{B} \in \mathcal{R}_S$, and any finite partial $\mathbf{P} \subseteq \mathbf{B}$, a finite $\mathbf{A} \in \mathcal{R}_S$, into which $\mathbf{P}$ can be embedded. Let $\mathbf{B}$ and $\mathbf{P}$ be as above. Our construction of a finite algebra into which $\mathbf{P}$ will be embeddable is to be carried out in two stages.

First, observe that since $\mathbf{B}$ is simple, for any $x \in B$ if $x < 1$ there exists a positive integer m such that $x^m = 0$. We will show that we can always embed $\mathbf{P}$ into an $\mathbf{B}$ for which the m above is uniform, i.e., there exists a positive integer m that $x^m = 0$ for any $x < 1$. This is equivalent to saying that $\mathbf{B}$ can always be so chosen that it satisfies EM_m, for some m.

For any residuated lattice $\mathbf{B}$, let $\mathbf{B}^-$ stand for its $\{\wedge, \vee, \cdot\}$-reduct. For an element $a \in B \setminus \{1\}$, $\mathbf{B}_a^-$ denotes the algebra $\big\langle (a] \vee \{1\}; \wedge, \vee, \cdot, 0, 1 \big\rangle$, where $(a]$ denotes the set $\{x : x \leq a\}$. Clearly, $\mathbf{B}_a^-$ is a subalgebra of $\mathbf{B}^-$.

Lemma 2.5. *For any $a, b, c \in B$, the set $U = \{x \leq a : x \cdot b \leq c\}$ has the greatest element equal to $a \wedge (b \to c)$.*

Proof. We have both $a \wedge (b \to c) \leq a$ and $(a \wedge (b \to c)) \cdot b \leq c$, clearly. Now suppose that $z \leq a$ and $z \cdot b \leq c$ hold. Then, $z \leq a \wedge (b \to c)$. Thus we have our lemma.

Let $\mathbf{B}$ be any simple residuated lattice, and $a \in B \setminus \{1\}$. We will define a binary operation $\xrightarrow{*}$ on $\mathbf{B}_a^-$, by putting:

$$x \xrightarrow{*} y = \begin{cases} a \wedge (x \to y), & \text{if } x \not\leq y, \\ 1, & \text{otherwise.} \end{cases}$$

By Lemma 2.5, the operation $\xrightarrow{*}$ is well-defined. Let us call the resulting algebra $\mathbf{B}_a^+$. In what follows, we will refer to the universe of either $\mathbf{B}_a^-$ or $\mathbf{B}_a^+$, by B_a. This is unambiguous, as they share the same universe.

Lemma 2.6. *$\mathbf{B}_a^+$ is a simple residuated lattice. Moreover, EM_k is valid in $\mathbf{B}_a^+$ for some k.*

Proof. To show that $\mathbf{B}_a^+$ is a residuated lattice, we only have to check whether, for any $x, y \in B_a$, the element $x \xrightarrow{*} y$ is the largest among all $z \in B_a$ with $z \cdot x \leq y$. This was already ensured by Lemma 2.5.

To show that $\mathbf{B}_a^+$ is simple, take any element $x \in B_a$ such that $x < 1$. Then, $x \leq a$. As $\mathbf{B}$ is simple, we have that for some $k < \omega$, $a^k = 0$ in $\mathbf{B}$. Thus, $x^k \leq a^k = 0$. Since the fusion in $\mathbf{B}_a^+$ coincides with the fusion in $\mathbf{B}$, $x^k = 0$ also in $\mathbf{B}_a^+$. Hence $\mathbf{B}_a^+$ is simple and EM_k is valid in $\mathbf{B}_a^+$.

Now, let $\mathbf{P}$ be the original partial algebra, and $\mathbf{B}$ the algebra into which $\mathbf{P}$ is embedded. We have:

Lemma 2.7. *There exists an algebra* $\mathbf{C} \in \mathcal{R}_S$ *such that* $\mathbf{C} \supseteq \mathbf{P}$ *and* $\mathbf{C}$ *satisfies* EM_k*, for some positive integer* k*.*

Proof. Define P_0 to be $P \setminus \{1\}$. As P_0 is finite, the join $\bigvee P_0$ is well-defined in $\mathbf{B}$, although not necessarily in $\mathbf{P}$. Since $\mathbf{B}$ is simple (hence subdirectly irreducible), $\bigvee P_0 < 1$ in $\mathbf{B}$, for otherwise, by Proposition 1.4, for some $p \in P_0$, $p = 1$, which is a contradiction.

Thus, the element $c = \bigvee P_0$ is different from 1. Now, consider $\mathbf{B}_c^+$ defined as above. By Lemma 2.6, EM_k is valid in $\mathbf{B}_c^+$ for some k. Observe also that, for any $a, b \in P$, and $\star \in \{\vee, \wedge, \cdot, \rightarrow\}$, if $a \star b$ is defined in P, then $a \star b$ in $\mathbf{P}$ is equal to $a \star b$ in $\mathbf{B}_c^+$. This is trivial for all operations except $\rightarrow$. However, if $a \rightarrow b$ is defined, then $a \rightarrow b$ belongs to P, and therefore it is either equal to 1 or smaller than c in $\mathbf{B}$. Thus, it remains unaltered in $\mathbf{B}_c^+$. Therefore, $\mathbf{B}_c^+$ may be taken as the algebra $\mathbf{C}$ whose existence was required by the lemma.

Before entering the second stage of our construction, we will define an auxiliary notion. Let us call the algebra $\mathbf{W} = \langle W; \vee, \cdot, 0, 1\rangle$ a *bounded, commutative, semilattice-ordered monoid* (or, for short, a *bocsoid*, although we do not particularly like this acronym) iff:

1. $\langle W; \vee, 0, 1\rangle$ is a join-semilattice with the greatest element 1 and the smallest element 0,
2. $\langle W; \cdot, 1\rangle$ is a commutative monoid satisfying: $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$.

Lemma 2.8. *Any finite bocsoid is a reduct of a residuated lattice.*

Proof. Let $\mathbf{W}$ be a finite bocsoid. Define the following two operations on its universe W:

- $x \wedge y = \bigvee\{z \mid z \leq x \ \&\ z \leq y\}$,
- $x \Rightarrow y = \bigvee\{w \mid w \cdot x \leq y\}$.

Under these definitions $\langle W; \vee, \wedge, \cdot, \Rightarrow, 0, 1\rangle$ becomes a residuated lattice, as it is straightforward to check. Note that if $\cdot$ is idempotent, then we get the well-known fact that every finite distributive lattice is a Heyting algebra.

Let us call a bocsoid $\mathbf{W}$ *k-potent* iff there is a positive integer k such that $\mathbf{W}$ satisfies $x^{k+1} = x^k$.

Lemma 2.9. *All k-potent bocsoids are locally finite.*

Proof. Let $\mathbf{W}$ be a k-potent bocsoid, and Z be a finite subset of W. By distributivity of $\cdot$ over $\vee$ each element generated by Z is of the form $\bigvee_{i=0}^{j-1} z_i$, for some positive integer j, where each z_i is either 1, or 0, or a fusion of some elements from Z.

As fusion is associative and commutative, we can dispense with parentheses and write each z_i as $a_0^{n_0} \ldots a_{l-1}^{n_{l-1}}$, with $a_0, \ldots, a_{l-1} \in Z$. Since $x^{k+1} = x^k$ holds, for a certain fixed k, the exponents in this expression are bounded from above by k, i.e., if $n_j > k$, then $a_0^{n_0} \ldots a_j^{n_j} \ldots a_{l-1}^{n_{l-1}} = a_0^{n_0} \ldots a_j^{k} \ldots a_{l-1}^{n_{l-1}}$. It follows, that an $X \subseteq W$, with $X = \{a_0, \ldots, a_{l-1}\}$, can generate, by means of fusion alone, at most $(k+1)^l + 1$ distinct elements. Thus, as there are only finitely many subsets of Z, the closure of Z under fusion is finite. As semilattices are locally finite, the closure of the latter under join is finite as well. This proves the lemma.

Theorem 2.2. *The class $\mathcal{R}_S$ of simple residuated lattices has the finite embeddability property.*

Proof. We have to show that every finite partial algebra $\mathbf{P}$ embeddable in $\mathcal{R}_S$ can be embedded into a finite algebra from $\mathcal{R}_S$. From Lemma 2.7 we know that there is a simple residuated lattice $\mathbf{C}$ satisfying $x^{k+1} = x^k$, for some positive integer k, and such that $\mathbf{C} \supseteq \mathbf{P}$. In fact we can take $\mathbf{C} = \mathbf{B}_{\mathbf{c}}^+$, where the latter has been defined in the proof of Lemma 2.7. Let also P_0 and c be the same as in that proof.

Let $\mathbf{C}^-$ be the $\{\vee, \cdot, 0, 1\}$-reduct of $\mathbf{C}$. Thus, $\mathbf{C}^-$ is a bocsoid. Let $\mathbf{W}$ be the sub-bocsoid of $\mathbf{C}^-$ generated by P. By the properties of $\mathbf{C}$, the bocsoid $\mathbf{C}^-$ is k-potent, indeed, it satisfies:

$$x^k = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases} \tag{simp}$$

because, by Lemma 2.7, so does $\mathbf{C}$. As P is finite, it follows, by Lemma 2.9, that so is $\mathbf{W}$. Now, let $\mathbf{A}$ be the residuated lattice resulting from endowing $\mathbf{W}$ with the operations defined in the proof of Lemma 2.8. Clearly, $\mathbf{A}$ is a finite residuated lattice; since $\mathbf{W}$ satisfies (simp), $\mathbf{A}$ is simple.

All that remains is to show that $\mathbf{P}$ is indeed a partial subalgebra of $\mathbf{A}$. There are two operations, for which something could go wrong, the ones that have been defined anew: $\wedge$ and $\rightarrow$. However, if $x, y \in P$ and $x \wedge y$, or $x \rightarrow y$, is defined in P, then, it is not difficult to verify from the definitions that the original operations coincide with the new ones. This finishes the proof.

Let $\mathcal{E}_k$ stand for the variety of k-potent residuated lattices, defined relative to $\mathcal{R}$ by the identity: $x^k = x^{k+1}$. Thus, $\mathcal{E}_k$ corresponds to the logic $\mathbf{FL_{ew}}[\mathrm{E}_k]$. Analogously, let $\mathcal{EM}_k$ stand for the variety of k-potent residuated lattices, defined relative to $\mathcal{R}$ by the identity: $x \vee \neg x^k = 1$; this corresponds to $\mathbf{FL_{ew}}[\mathrm{EM}_k]$.

Theorem 2.3. *Both $\mathcal{E}_k$ and $\mathcal{EM}_k$ have the finite embeddability property.*

Proof. Let $\mathbf{P}$ be a partial subalgebra of an algebra $\mathbf{C}$ from $\mathcal{E}_k$ ($\mathcal{EM}_k$). Let $\mathbf{C}^-$ and $\mathbf{W}$ be as in the proof of Theorem 2.2. Then, $\mathbf{W}$ is a finite bocsoid satisfying $x^k = x^{k+1}$ ($x \vee \neg x^k = 1$), and the rest of the proof applies without any changes.

Conclusions, questions and other remarks

Let us start with proving the main theorem of this chapter.

Theorem 2.4. *The variety $\mathcal{R}$ of residuated lattices is generated by its finite simple members.*

Proof. It suffices to show that for any non-theorem α of $\mathbf{FL_{ew}}$ there is a finite simple residuated lattice $\mathbf{A}$ falsifying α. By Theorem 2.1, there exist a simple residuated lattice $\mathbf{B}$, and a valuation v on B, with $v(\alpha) < 1$. Let Σ be the set of all subformulas of α. Take $P = v(\Sigma) \vee \{0, 1\}$, and define a partial algebra $\mathbf{P}$ putting, for $\star \in \{\wedge, \vee, \rightarrow, \cdot\}$, $v(\beta) \star v(\gamma) = v(\beta \star \gamma)$, if $\beta \star \gamma$ belongs to Σ, and leaving it undefined otherwise. Thus defined, $\mathbf{P}$ is a finite partial algebra from $\mathcal{R}_S$, and $\mathbf{P} \subseteq \mathbf{B}$. The construction from the previous section yields a finite simple residuated lattice $\mathbf{A}$, into which $\mathbf{P}$ can also be embedded. For a variable x, let us put $w(x) = v(x)$, if x occurs in α, and be arbitrary otherwise. Extending w to a valuation, and, as usual, retaining the same symbol for it, we obtain, $w(\alpha) < 1$ in $\mathbf{A}$, which ends the proof.

Clearly, it does not follow that $\mathcal{R}$ is semisimple (a variety is semisimple if all its si members are simple). Consider the chains of varieties $\mathcal{E}_n$ and $\mathcal{EM}_n$ $(n \in \omega)$, defined in Chapter 1. We have:

Theorem 2.5. *The varietal joins $\bigvee_{n\in\omega} \mathcal{E}_n$ and $\bigvee_{n\in\omega} \mathcal{EM}_n$ are both equal to $\mathcal{R}$.*

Proof. Since $\bigvee_{n\in\omega} \mathcal{E}_n \supseteq \bigvee_{n\in\omega} \mathcal{EM}_n$, we only have to show that any non-theorem can be falsified on an algebra that satisfies:

$$x^k = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

for some positive integer k. The algebra $\mathbf{A}$ from the previous proof is such.

Corollary 2.3. *The intersections of the logics $\mathbf{FL_{ew}}[\mathrm{E}_n]$ and $\mathbf{FL_{ew}}[\mathrm{EM}_n]$for $n \in \omega$ is equal to $\mathbf{FL_{ew}}$.*

Several conclusions about the varieties $\mathcal{E}_n$ and $\mathcal{EM}_n$ follow immediately from finite embeddability (Theorem 2.3). For example, the varieties $\mathcal{EM}_n$ are all generated by their finite simple members, and the varieties $\mathcal{E}_n$ are all generated by their finite (not necessarily simple) members. It follows that the logics $\mathbf{FL_{ew}}[\mathrm{EM}_n]$, and $\mathbf{FL_{ew}}[\mathrm{E}_n]$, all have finite model property, and thus are decidable.

We finish off with some questions. The first concerns possible analogues of our result for other varieties of residuated lattices. Although, as the situation within the variety of Heyting algebras suggests, any thorough characterisation of subvarieties of $\mathcal{R}$ generated by their finite members is rather elusive, we would like to ask:

Question 2.1. *Which (well-known) subvarieties of $\mathcal{R}$ are generated by their finite/simple/finite and simple members?*

We do not even know whether the variety of classical residuated lattices is generated by its finite simple members, although this seems a plausible conjecture, as it is generated both by its finite members and by its simple members. Other natural candidates to consider are: linear residuated lattices, distributive residuated lattices. The obstacle with transferring our proof to these seems to lie in the fact that where the proof theoretic part works well, the algebraic part apparently fails, and *vice-versa.* Take classical residuated lattices, for instance. There, cut-elimination holds, thus, the proof theoretic part works and we have generation by simple members (this is precisely the content of Grišin's proof that inspired us). Yet, the algebraic construction fails, at least in its simplest form. On the other hand, in linear, or distributive case, the algebraic construction carries through, but it is not clear whether there are chances for a Gentzen system with cut-elimination or, at the very least, subformula property. This leads to the following:

Question 2.2. *Prove semisimplicity of free residuated lattices semantically.*

Such a proof, may (or may not, to be sure) be easier to transfer to subvarieties of $\mathcal{R}$.

Among residuated structures a prominent place occupy *partially ordered commutative residuated integral monoids*, commonly referred to as *pocrims* (cf. [10] for more about pocrims). The class $\mathcal{P}_k$ of k-potent (i.e., satisfying $x^{k+1} = x^k$; some authors prefer to call these $k+1$-potent) pocrims is a variety, for each positive integer k. After [3], we ask:

Question 2.3. *Does $\mathcal{P}_k$ have the finite embeddability property for $k > 1$?*

For $k = 1$ the answer is affirmative and easy. For it can easily be checked that the technique from [43] embeds any idempotent pocrim into an idempotent residuated lattice. This actually only restates the well-known fact that idempotent pocrims are precisely Brouwerian semilattices. Thus, if $\mathbf{P}$ is a finite partial subalgebra of an algebra $\mathbf{A} \in \mathcal{P}_1$, then it is also a partial subalgebra of an algebra $\mathbf{B}$ from $\mathcal{E}_1$. Thus, by Theorem 2.3, $\mathbf{P}$ is finitely embeddable in $\mathcal{E}_1$; hence, by taking the appropriate reduct, also in $\mathcal{P}_1$. This simple reasoning, however, fails for $k > 1$, for whereas it remains true that any k-potent pocrim can be embedded into a residuated lattice, the latter may in general fail to be k-potent.

3. SPLITTINGS

Splittings have proved to be a useful tool in analysing many lattices of logics, especially intermediate and modal logics. Defined in abstract lattice-theoretic terms, splittings, especially if many of them exist, witness a tractable structure of the lattice in question. Thus, the usefulness of *splitting methods* as they came to be known, depends on the existence of many splittings, and the more splittings there are, the better they work. In this chapter, which is based on our [37], we prove that $L^v(\mathcal{R})$ fares pretty badly in this respect, there being only one, somewhat trivial, splitting.

A pair $(\mathcal{Q}_1, \mathcal{Q}_2)$ of sub(quasi)varieties of a given (quasi)variety $\mathcal{W}$ is said to *split* $L^v(\mathcal{W})$ $(L^q(\mathcal{W}))$ iff $\mathcal{Q}_1 \not\subseteq \mathcal{Q}_2$ and for any $\mathcal{S}$, sub(quasi)variety of $\mathcal{W}$, either $\mathcal{Q}_1 \subseteq \mathcal{S}$ or $\mathcal{S} \subseteq \mathcal{Q}_2$. In other words, $\mathcal{Q}_2$ is the largest sub(quasi)variety of $\mathcal{W}$ not containing $\mathcal{Q}_1$.

If $(\mathcal{V}_1, \mathcal{V}_2)$ is a *splitting pair* of subvarieties of a variety $\mathcal{V}$, i.e., when $(\mathcal{V}_1, \mathcal{V}_2)$ splits the lattice $L^v(\mathcal{V})$ of subvarieties of $\mathcal{V}$, then $\mathcal{V}_1$ is generated by a si algebra, called *splitting algebra*. To avoid confusion we will speak of splitting algebras only within the context of splittings in the lattice of subvarieties of a given variety, and never refer by this name to algebras that split the lattice of subquasivarieties of that variety. Thus, it may (and will) happen that an algebra that splits the lattice of subquasivarieties of a variety $\mathcal{V}$ is not a splitting algebra in $\mathcal{V}$ in our technical usage.

If $\mathcal{V}$ is congruence-distributive and generated by its finite members, then every splitting algebra in $\mathcal{V}$ is finite and uniquely determined by the splitting pair (cf. [39]). If $\mathcal{V}$ is of finite type and has EDPC, then every finite si member of $\mathcal{V}$ is splitting (cf.[5]). The next fact follows easily from what has just been recalled.

Proposition 3.1. *For any $n \in \omega$, all finite si algebras in $\mathcal{E}_n$ split $L^v(\mathcal{E}_n)$.*

Since every finite si algebra splits the lattice of subquasivarieties containing it, we have the following fact that provides a contrasting background for the things to come.

Proposition 3.2. *All finite si residuated lattices split $L^q(\mathcal{R})$.*

Finally, let us state the following easy observation:

Proposition 3.3. *The two-element boolean algebra $\mathbf{2}$ splits $L^v(\mathcal{R})$.*

Proof. Since $\mathbf{2}$ is a subalgebra of any nontrivial residuated lattice, the largest subvariety of $\mathcal{R}$ not containing $\mathcal{V}(\mathbf{2})$ is the trivial subvariety $\mathcal{T}$. Thus, $(\mathcal{V}(\mathbf{2}), \mathcal{T})$ is the relevant splitting pair.

Yankov terms

We will employ (a version of) the technique introduced by Yankov in [58]. Let $\mathbf{A}$ be a finite si residuated lattice. Fix a set X of $|A|$ distinct variables, and index them by the elements of A, so that x_a, x_b are distinct iff $a \neq b$. Let $\neg$ be the term operation defined as $\neg z = z \to 0$. The *diagram* of $\mathbf{A}$ is defined, as usual, by $\Delta_{\mathbf{A}} = \bigwedge\{x_{\neg a} \leftrightarrow \neg x_a : a \in A\} \wedge \bigwedge\{x_{a \diamond b} \leftrightarrow x_a \diamond x_b : a, b \in A, \diamond \in \{\vee, \wedge, \cdot, \to\}\}$.

Then, the *Yankov term of order n* for $\mathbf{A}$ is defined as $Y^{(n)}_{\mathbf{A}} = \Delta^n_{\mathbf{A}} \to x_\star$, where $\star \in A$ is the smallest member of the congruence filter F_μ, i.e., the congruence class $1/\mu$, with μ being the monolith of $\mathbf{A}$. We will call F_μ the *monolithic* filter.

Given a residuated lattice $\mathbf{A}$, a *valuation* in $\mathbf{A}$ (with $\mathbf{A}$ often not mentioned explicitly, if it is clear from context) will mean, as usual, any homomorphism from the absolutely free algebra of the type of residuated lattices into $\mathbf{A}$.

Lemma 3.1. *Let $\mathbf{B} \in \mathcal{R}$. Then, $\mathbf{A} \subseteq \mathbf{B}$ iff there is a valuation v, such that $\mathbf{B} \models_v \Delta_{\mathbf{A}} = 1$ & $x_\star \neq 1$.*

Proof. For the "only if" part, observe that $\mathbf{A} \models_v \Delta_{\mathbf{A}} = 1$ & $x_\star \neq 1$ under the natural valuation $v(x_a) = a$. Since $\mathbf{A} \subseteq \mathbf{B}$, so does $\mathbf{B}$ (later on, as here, we will frequently use "$\subseteq$" to denote the relation of being a subalgebra, relying on context to distinguish between different meanings of the symbol).

For the "if" part, suppose we have a valuation v with the required properties. Then, define a map $h : \mathbf{A} \longrightarrow \mathbf{B}$, putting $h(a) = v(x_a)$. This map, clearly, is a homomorphism into $\mathbf{B}$. Moreover, we have $h(1) = v(x_1) = 1^{\mathbf{B}} \neq v(x_\star) = h(\star)$. Therefore, as $\mathbf{A}$ is si and every nontrivial congruence on $\mathbf{A}$ contains $(1, \star)$, the kernel of h must be the trivial congruence on $\mathbf{A}$. Hence, h is an embedding.

We have seen that the two-element boolean algebra splits $L^v(\mathcal{R})$. The rest of the paper is devoted to proving that no other algebra has this property. The key to our proof technique is contained in the lemma below.

Lemma 3.2. *The following are equivalent:*

(i) *$\mathbf{A}$ is not a splitting algebra in $\mathcal{R}$,*

(ii) *$\forall i \in \omega\ \exists \mathbf{B} \in \mathcal{R} : \mathbf{A} \notin \mathcal{V}(\mathbf{B})$ and $\mathbf{B} \not\models Y^{(i)}_{\mathbf{A}} = 1$.*

Proof. To prove the implication from (ii) to (i) take a sequence of algebras $\mathbf{B}_i$, $(i \in \omega)$ such that $\mathbf{B}_i \not\models Y^{(i)}_{\mathbf{A}} = 1$ and $\mathbf{A} \notin \mathcal{V}(\mathbf{B}_i)$. Let $k = |A|$. Choose a sequence $\overline{b(i)} = \langle b(i)_0, \dots, b(i)_{k-2}, s(i)\rangle$ of elements of $\mathbf{B}_i$ such that $Y^{(i)}_{\mathbf{A}}(b(i)_0, \dots, b(i)_{k-2}, s(i)) < 1$ in $\mathbf{B}_i$. Thus, $\Delta^i_{\mathbf{A}}(b(i)_0, \dots, b(i)_{k-2}, s(i)) \not\leq s(i)$. Take $\mathbf{B} = \prod_{i\in\omega} \mathbf{B}_i$, and consider the element $\overline{b} = \big\langle \langle b(i)_0 : i \in \omega\rangle, \dots, \langle b(i)_{k-2} : i \in \omega\rangle, \langle s(i) : i \in \omega\rangle \big\rangle \in B^k$. Let s stand for $\langle s(i) : i \in \omega\rangle$. It is readily seen that $\forall i \in \omega : \Delta^i_{\mathbf{A}}(\overline{b}) \not\leq s$ in $\mathbf{B}$. Thus, the congruence filter F generated by $\Delta_{\mathbf{A}}(\overline{b})$ does not contain s. Hence, taking the congruence θ determined by F, we obtain that $\Delta_{\mathbf{A}}(\overline{b}/\theta) = 1$ and $s/\theta \neq 1$ in the quotient $\mathbf{B}/\theta$. By Lemma 1(i), it yields $\mathbf{A} \subseteq \mathbf{B}/\theta$; whence, $\mathbf{A} \in \mathcal{V}(\mathbf{B})$. Now, if $\mathbf{A}$ were splitting, there would exist the largest subvariety $\mathcal{V}$ of $\mathcal{R}$ not containing $\mathcal{V}(\mathbf{A})$. Then, however, $\mathcal{V}$ would contain all the varieties $\mathcal{V}(\mathbf{B}_i)$ $(i \in I)$. In particular, $\forall i \in I : \mathbf{B}_i \in \mathcal{V}$, and thus $\mathbf{B} \in \mathcal{V}$. Hence, $\mathbf{A} \in \mathcal{V}$; a contradiction which finishes this part of the proof.

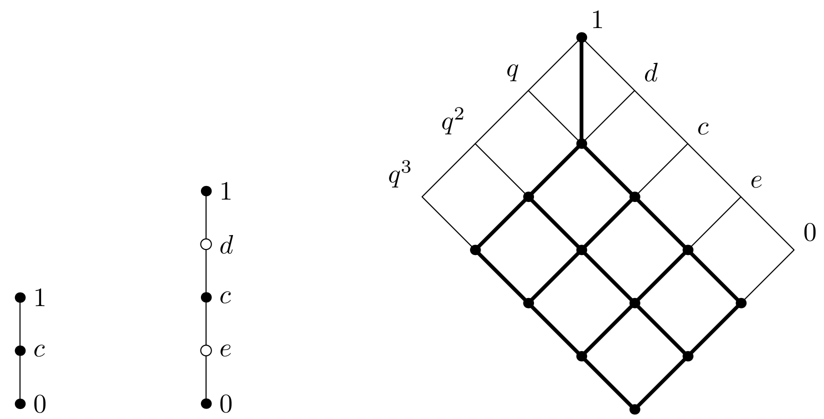
To show that (i) implies (ii) we will argue the contrapositive. Assume that $\exists i \in \omega\ \forall \mathbf{B} \in \mathcal{R} : \mathbf{A} \notin \mathcal{V}(\mathbf{B})$ implies $\mathbf{B} \models Y^{(i)}_{\mathbf{A}} = 1$. Let m be the smallest with this property. We will show that the subvariety $\mathcal{W}$ of $\mathcal{R}$ defined by the identity $Y^{(m)}_{\mathbf{A}} = 1$ is the largest such that it does not contain $\mathcal{V}(\mathbf{A})$. Obviously, $\mathcal{V}(\mathbf{A}) \not\subseteq \mathcal{W}$, as otherwise we would have $\mathbf{A} \models Y^{(m)}_{\mathbf{A}} = 1$ which cannot be the case. Take any subvariety $\mathcal{V}$ of $\mathcal{R}$, with

$\mathbf{A} \notin \mathcal{V}$. Let $\mathbf{F}$ be the free countably generated algebra in $\mathcal{V}$. Thus, $\mathcal{V} = \mathcal{V}(\mathbf{F})$. This, by our assumption, implies that $\mathbf{F} \models Y_{\mathbf{A}}^{(m)} = 1$. Thus, $\mathcal{V} \models Y_{\mathbf{A}}^{(m)} = 1$ and thus $\mathcal{W} \supseteq \mathcal{V}$ as desired.

Now, to make use of Lemma 3.2 in showing that a certain residuated lattice $\mathbf{A}$ does not split $L^v(\mathcal{R})$, we have to exhibit, for any given $i \in \omega$, another residuated lattice $\mathbf{B}$ with the properties required by (ii). The proof will proceed by construction. Taking $\mathbf{A}$ as a starting point, we will first construct, for each $i \in \omega$, a certain large residuated lattice $\mathbf{C}$, which falsifies $x^i = x^{i+1}$. The algebra $\mathbf{C}$, however, will contain $\mathbf{A}$ as a subalgebra, so we will "distort" it by means of another construction and arrive at an algebra we want.

An example

To illustrate our constructions, we present a simple concrete example. Suppose we want to show that the three-element Łukasiewicz algebra $\mathbf{Ł}_3 = \langle\{1, c, 0\}; \wedge, \vee, \cdot, \to, 0, 1\rangle$ with $0 < c < 1$, $c^2 = 0$ (these conditions determine all the operations uniquely), does not split $L^v(\mathcal{R})$. We will produce an algebra with the desired properties for $i = 2$. We start with "expanding" L_3, by adding two new elements: d and e, and stipulating $0 < e < c < d < 1$. Further, we set $d^2 = c$, $d^3 = e$, $d^4 = 0$. Then, we set all the other fusions so that the resulting structure becomes the five-element Łukasiewicz algebra $\mathbf{Ł}_5$. Clearly, $\mathbf{Ł}_3 \subseteq \mathbf{Ł}_5$; moreover, $\mathbf{Ł}_5$ falsifies $x^2 = x^3$, verified by $\mathbf{Ł}_3$. Then, we proceed to distort $\mathbf{Ł}_5$ in a suitable way. To do this, we take the simple Łukasiewicz algebra $\mathbf{Ł}_{p+1}$, for the smallest prime p greater or equal to the number of elements in our original algebra. As the original algebra was $\mathbf{Ł}_3$, we take $\mathbf{Ł}_4$. Our target algebra will be called $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$. As its universe, we take a subset of the direct product $L_5 \times L_4$, namely $L_5 \odot L_4 = (L_5 \times L_4) \setminus (\{\langle 1, x\rangle : x \neq 1\} \cup \{\langle y, 1\rangle : y \neq 1\})$. All the operations, except residuation, are defined as they would have been in the direct product, coordinatewise. Residuation is then implicitly defined, by finiteness of the universe and distributivity of fusion over join. See the picture below for an intuitive idea, and the penultimate section of this chapter for the explicit definitions. We will follow the notational convention from there in naming the elements on the second coordinate by 1, q, q^2, $q^3 = 0$, and viewing the elements of $L_5 \odot L_4$ as elements from the first coordinate indexed by the elements from the second. Thus, for instance, d_q is the unique coatom of $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$, $(d_q)^2 = d^2_{q^2} = c_{q^2}$, etc. The algebras $\mathbf{Ł}_3$, $\mathbf{Ł}_5$, and $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$ are pictured below. In L_5 the white dots mark the new elements inserted between the elements of L_3. The grid on the right-hand side represents (the lattice structure of) $\mathbf{Ł}_5 \times \mathbf{Ł}_4$, the thick lines and black dots depict $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$.



As $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$ has unique coatom it is si. In fact, $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$ is even simple, for $(d_q)^4 = d^4_{q^4} = 0$. Now, if $\mathbf{Ł}_3$ belonged to $\mathcal{V}(\mathbf{Ł}_5 \odot \mathbf{Ł}_4)$, we would have $\mathbf{Ł}_3 \in SH(\mathbf{Ł}_5 \odot \mathbf{Ł}_4)$, by Jónsson Lemma and CEP for $\mathcal{R}$. Since $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$ is simple, $\mathbf{Ł}_3$ cannot be a subalgebra of any proper homomorphic image of $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$. Thus, the only possibility is $\mathbf{Ł}_3 \subseteq \mathbf{Ł}_5 \odot \mathbf{Ł}_4$. However, since $\mathbf{Ł}_{p+1}$ is one-generated, for any prime p, any purported embedding will produce at least 4 distinct elements in $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$. Hence, $\mathbf{Ł}_3 \notin \mathcal{V}(\mathbf{Ł}_5 \odot \mathbf{Ł}_4)$.

It remains to show that $\mathbf{Ł}_5 \odot \mathbf{Ł}_4$ falsifies $Y^{(2)}_{\mathbf{Ł}_3}$. Now, even though $\mathbf{Ł}_3$ has only three elements, its full diagram is a bit unwieldy. We will write down only those conjuncts which will not be equal to 1 under our intended valuation. These are: $x_0 \leftrightarrow \neg x_1$, $x_c \leftrightarrow \neg x_c$, and $x_1 \leftrightarrow \neg x_0$. Notice that only residuation is involved, and only in the disguise of negation (the former is generally true, the latter is a coincidence). Our intended valuation v is: $v(x_1) = 1_1$, $v(x_c) = c_q$, $v(x_0) = 0_q$. Under this valuation we have: $v(\Delta_{\mathbf{Ł}_3}) = v((x_0 \leftrightarrow \neg x_1) \wedge (x_c \leftrightarrow \neg x_c) \wedge (x_1 \leftrightarrow \neg x_0)) = d_{q^2} \wedge d_{q^2} \wedge d_q = d_{q^2}$. Thus, $v(\Delta_{\mathbf{Ł}_3}) = d_{q^2}$, and $v(\Delta^2_{\mathbf{Ł}_3}) = (d_{q^2})^2 = d_0^2 = c_0$. On the other hand, $\star = 0$ in $\mathbf{Ł}_3$, and thus $v(x_\star) = 0_q$. Therefore, $v(Y^{(2)}_{\mathbf{Ł}_3}) \neq 1$.

In the example above, we were very lucky at the first stage of the construction. It turned out, that inserting an extra element between x and cx, for all $x \in \{1, c, 0\}$ with $cx < x$, produced a suitable residuated lattice. Unfortunately, this cannot work in general. There are simple examples of residuated lattices for which this will destroy the lattice structure, or residuation (or both). We will need something more refined. The next section deals with developing an appropriate construction.

On the other hand, the second stage always works exactly as in our example. Thus, its exposition in the section following the next will be rather brief.

Expansions

We begin the construction by fixing a finite si residuated lattice $\mathbf{A}$, with the coatom c. Now, take the set $A_0 = \{a \in A : ca < a\}$, and let D be any set disjoint from A, with $|D| = |A_0|$. Thus, by means of any bijection, we can index the elements of D by the elements of A_0, getting $D = \{d_a : a \in A_0\}$. Let $B = A \cup D$. We will proceed to define a relation and an operation on B.

Definition 3.1. *For $x, y \in B$, we put $x \leq y$ if either:*

- *$x, y \in A$ and $x \leq^{\mathbf{A}} y$; or*
- *$x = d_a \in D, y \in A$ and $a \leq^{\mathbf{A}} y$; or*
- *$x \in A, y = d_a \in D$ and $x \leq^{\mathbf{A}} ca$; or*
- *$x = d_a, y = d_b \in D$ and $a \leq^{\mathbf{A}} b$.*

Notice that we have $ca < d_a < a$ whenever $ca < a$.

Then, we pass on to define a binary operation '$\cdot$' on B. To avoid overloaded notation, we will abbreviate $x \cdot^{\mathbf{A}} y$ everywhere by xy. Thereby, we commit ourselves to never abbreviating the new operation $x \cdot y$, within the present section.

Definition 3.2. *For any $x, y \in B$, let:*

$$x \cdot y = y \cdot x = \begin{cases} xy, & \text{if } \; x, y \in A; \\ d_{ay}, & \text{if } \; x = d_a \in D, \; y \in A, \; cay < ay; \\ ay, & \text{if } \; x = d_a \in D, \; y \in A, \; cay = ay; \\ cab, & \text{if } \; x = d_a, y = d_b \in D. \end{cases}$$

Lemma 3.3. *The structure $\mathbf{B} = \langle B; \cdot, 1, 0, \leq \rangle$ is a partially ordered, bounded, commutative, integral monoid. Moreover, '$\cdot$' is monotonic, i.e., if $x \leq y$, then $z \cdot x \leq z \cdot y$, for any $x, y, z \in B$.*

Proof. Reflexivity of the relation $\leq$ is obvious. For antisymmetry, observe that if $x \in A$, $y \in D$, and $x \leq y$, or if $x \in D$, $y \in A$, and $x \leq y$, then $y \not\leq x$; in other cases it follows from the antisymmetry of $\leq^{\mathbf{A}}$. Transitivity for triples $x, y, z \in A$ or $x, y, z \in D$ is obvious as well. There remain six cases to consider, of which we take one as an example. Suppose $x = d_a \leq y \in A$, $y \leq z = d_b$. Then, $a \leq^{\mathbf{A}} y$, $y \leq^{\mathbf{A}} cb \leq^{\mathbf{A}} b$. By transitivity of $\leq^{\mathbf{A}}$, $a \leq^{\mathbf{A}} b$. By definition of $\leq$, $d_a \leq d_b$ as needed. Clearly, the top element 1 and the bottom element 0 of $\mathbf{A}$ remain the top and the bottom of $\mathbf{B}$ under the ordering $\leq$.

Commutativity of '$\cdot$' is clear from its definition. Its associativity is obvious for triples $x, y, z \in A$. With commutativity granted, the number of remaining cases reduces to three, of which we again take one as an example. Let $x = d_a, y \in A, z = d_b$. We have four cases:

(1) $cyb < yb$, $cay < ay$. Then, $d_a \cdot (y \cdot d_b) = d_a \cdot d_{yb} = cayb$. Starting from the other end, we get $(d_a \cdot y) \cdot d_b = d_{ay} \cdot d_b = cayb$, as needed.

(2) $cyb < yb$, $cay = ay$. Observe that the latter forces $cayb = ayb$. Then, $d_a \cdot (y \cdot d_b) = d_a \cdot d_{yb} = cayb$. From the other end, $(d_a \cdot y) \cdot d_b = ay \cdot d_b = ayb = cayb$, as needed.

(3) $cyb = yb$, $cay < ay$. The former forces $cayb = ayb$. Then, $d_a \cdot (y \cdot d_b) = d_a \cdot yb = ayb$; and $(d_a \cdot y) \cdot d_b = d_{ay} \cdot d_b = cayb = ayb$, as well.

(4) $cyb = yb$, $cay = ay$. Then $cayb = ayb$. Further, $d_a \cdot (y \cdot d_b) = d_a \cdot yb = ayb$; also $(d_a \cdot y) \cdot d_b = ay \cdot d_b = ayb$.

Since it readily seen that $1 \cdot x = x$ for all $x \in B$, the unit element of the monoid coincides with the top of the partial order, which is another way of saying that the monoid in question is integral. Thus, we have shown the first part of Lemma 3.3.

For the "moreover" part, we again take only one of eight possible cases. Suppose $A \ni x \leq y \in A$, and $D \ni z = d_a$. Then, out of another four cases, we pick the trickiest, namely, when $x \cdot d_a = ax$ and $y \cdot d_a = d_{ay}$. We must thus have, $cax = ax$ and $cay < ay$. As $x \leq y$, and '$\cdot^{\mathbf{A}}$' is monotonic, we get $ax = cax \leq cay < ay$. By definition, $cay < d_{ay}$, and thus, $x \cdot d_a = ax \leq cay < d_{ay} = y \cdot d_a$, as required.

So far, we have been dealing with two "sorts" of elements: members of A, and members of D. To get rid of this tiresome division, let's write d instead of d_1 (notice that the element d_1 indeed exists, i.e., is in D, for $c1 = c < 1$), and state:

Lemma 3.4. *For any $x, y \in A$, the following hold:*

(i) *if $cx < x$, then $d \cdot x = d_x$, otherwise $d \cdot x = x$;*

(ii) $d \cdot x \geq y$ *iff* $cx \geq y$*;*

(iii) $d \cdot x \cdot d \cdot y = cxy$*;*

(iv) $d \cdot x \leq d \cdot y$ *iff* $d \cdot x \leq y$ *iff* $x \leq y$*.*

Proof. The only point which is not an immediate consequence of the definitions is (iv). To prove it, notice first, that if $x \leq y$, then $d \cdot x \leq y$, and $d \cdot x \leq d \cdot y$, by monotonicity of '$\cdot$'. Now, if $d \cdot x \leq d \cdot y$, then, as $d \cdot y \leq y$, we get, $d \cdot x \leq y$. It remains to show that this forces $x \leq y$. Suppose $d \cdot x \leq y$. If $d \cdot x = d_x$, then $x \leq y$ follows by definition; otherwise, i.e., when $d \cdot x = x$, it follows trivially.

To state the next observation, it will be convenient to view **B** as a partial algebra $\langle B; \wedge^?, \vee^?, \to^?, \cdot, 0, 1\rangle$, with the "questionable" operations only partially defined. To be explicit, let us say that $\wedge^?, \vee^?, \to^?$ coincide, respectively, with the meet, join, and residuation, whenever they exist, and are undefined otherwise.

Lemma 3.5. **A** *is a subalgebra of* **B**.

Proof. Obviously, the $\{\cdot, 0, 1\}$-reduct of **A** is a subalgebra of (the $\{\cdot, 0, 1\}$-reduct of) **B**. It remains to verify the same for the three "questionables".

Take any $a, b \in A$. We claim that $a \wedge^{\mathbf{A}} b$, and $a \vee^{\mathbf{A}} b$ are, respectively, the infimum and supremum of $\{a, b\}$ in **B**. If a z from A has $z \leq a$, $z \leq b$, then, clearly, $z \leq a \wedge^{\mathbf{A}} b$. The dual argument works for join. Take now an $s \in D$ with $s \leq a$, $s \leq b$. We have, $s = d \cdot e$, for some $e \in A$, thus, by Lemma 3.4(iv), $e \leq a$, $e \leq b$. As $e \in A$, we get $e \leq a \wedge^{\mathbf{A}} b$. Thus, $s = d \cdot e \leq a \wedge^{\mathbf{A}} b$, as required. Then, suppose $s \in D$ and $s \geq a$, $s \geq b$. As previously, $s = d \cdot e$ with $e \in A$. By Lemma 3.4(ii) then, $ce \geq a$ and $ce \geq b$, from which it follows that $ce \geq a \vee^{\mathbf{A}} b$ and, thus, by Lemma 3.4(ii) again, $d \cdot e = s \geq a \vee^{\mathbf{A}} b$ as needed. Therefore, $a \wedge^{\mathbf{A}} b = a \wedge^? b$, and $a \vee^{\mathbf{A}} b = a \vee^? b$.

Next, we claim that $a \to^{\mathbf{A}} b$ is the residual of $\langle a, b\rangle$. We have to show that $s \cdot a \leq b$ iff $s \leq a \to^{\mathbf{A}} b$. This is obvious, when $s \in A$. Suppose $s \notin A$, i.e., $s \in D$, and thus, $s = d \cdot e$, for some $e \in A$. As an easy consequence of Definition 3.2, we have $d \cdot e \cdot a = d \cdot ea$. Thus, we get: $s \cdot a = d \cdot ea \leq b$ iff $ea \leq b$ iff $e \leq a \to^{\mathbf{A}} b$ iff $d \cdot e = s \leq a \to^{\mathbf{A}} b$, where the first and the last equivalence follow by Lemma 3.4(iv). Therefore, $a \to^{\mathbf{A}} b = a \to^? b$, as required.

Despite Lemma 3.5 above, we cannot expect **B** to be a fully fledged (i.e., not partial) residuated lattice. Indeed, simple examples show that **B** might be neither residuated nor a lattice. To remedy this unwelcome situation, we will resort to a fairly standard method of "completing" a poset to obtain a (complete) lattice. There are at least two such well-known completions, namely, the Dedekind-McNeille completion, or completion by cuts (see, e.g., [43] or [47] for uses of this completion in the context of logics without contraction), and the Alexandroff completion, or completion by lower ends. Without getting into details, let us say that neither of the two works in this case. Completion by cuts, while retaining the lattice structure of **A**, is too tight to provide residuation. On the other hand, completion by lower ends, while providing residuation, is too loose to preserve the lattice structure of **A**. Fortunately, these two drawbacks cancel each other out at some point. This point we are going to define now.

Definition 3.3. *A subset X of B is closed, if the following four conditions are satisfied:*

- $0^{\mathbf{A}} \in X$;
- $\forall x, y \in B : x \in X$ *and* $y \leq x$ *imply* $y \in X$;
- $\forall x, y \in A : x \in X$ *and* $y \in X$ *imply* $x \vee y \in X$;
- $\forall x, y \in A : d \cdot x \in X$ *and* $d \cdot y \in X$ *imply* $d \cdot (x \vee y) \in X$.

Since, as it is easy to verify, the intersection of any family of closed sets in the above sense is itself closed, we can define $C : \wp(B) \longrightarrow \wp(B)$ to be the map sending each $X \subseteq B$ to the smallest closed subset of B containing X. As usual we denote it by $C(X)$ and call the closure of X. To justify this terminology, we have the following fact, with an entirely straightforward proof.

Lemma 3.6. *The map C defined above is a closure operation on B.*

For a closed $X \subseteq B$, define $\hat{x}$ to be $\bigvee\{z \in A : z \in X\}$, and $\dot{x}$ to be $\bigvee\{z \in A : d \cdot z \in X\}$. The joins here are taken in B, but, by Lemma 3.5 and finiteness, joins of elements of A exist, and are again in A. Thus, the definitions are legitimate.

Lemma 3.7. *If $X \subseteq B$ is closed, then $X = (\hat{x}] \cup (d \cdot \dot{x}]$. Moreover, if $a \in A$, then both $(a]$ and $(d \cdot a]$ are closed.*

Proof. For the first statement, observe that we have $\hat{x} \in A$ and $\dot{x} \in A$. By closedness of X, $\hat{x} \in X$. Now, $d \cdot \dot{x} = d \cdot (\bigvee\{x \in A : d \cdot x \in X\}) = d \cdot (x_0 \vee \ldots \vee x_{k-1})$, for some $k \in \omega$. Further, $d \cdot x_0, \ldots, d \cdot x_{k-1} \in X$. From this, again, by closedness of X, it follows that $d \cdot \dot{x} \in X$. This proves $X \supseteq (\hat{x}] \cup (d \cdot \dot{x}]$.

To show the other inclusion, notice firstly that if $z \in X|_A$, then $z \leq \hat{x}$. Secondly, if $z \in X \setminus A$, then $z = d \cdot y$, for some $y \in A$, and $y \leq \dot{x}$. Thus, $z = d \cdot y \leq d \cdot \dot{x}$. Altogether, $z \in (\hat{x}] \cup (d \cdot \dot{x}]$, as needed.

Then, to see that $(a]$ is closed, we only need to check that if $d \cdot x,\ d \cdot y \leq a$, then $d \cdot (x \vee y) \leq a$, the three other conditions being trivially satisfied by $(a]|_A$ being an ideal of $\mathbf{A}$. Since $a \in A$, we have $x, y \leq a$, hence, $x \vee y \leq a$. Thus, $d \cdot (x \vee y) \leq a$.

Finally, to show that $(d \cdot a]$ is closed, we notice that $(d \cdot a]|_A$ is also an ideal of $\mathbf{A}$, thus, we only have to verify the fourth condition. If $d \cdot x$ and $d \cdot y$ are in $(d \cdot a]$, for some $x, y \in A$, then $x \vee y \leq a$; hence, $d \cdot (x \vee y) \leq d \cdot a$, i.e., $d \cdot (x \vee y) \in (d \cdot a]$ as required.

For $X, Y \subseteq B$, let $X \Rightarrow Y$ be $\{z \in B \mid \forall x \in X : z \cdot x \in Y\}$, and $X \circ Y$ be $\{x \cdot y \mid x \in X, y \in Y\}$.

Lemma 3.8. *Let X, Y be closed subsets of B, and let $Q = C(X \circ Y)$. The following hold:*

(i) $\hat{q} = c\dot{x}\dot{y} \vee \hat{x}\hat{y}$, $\dot{q} = \dot{x}\hat{y} \vee \hat{x}\dot{y}$; *therefore* $C(X \circ Y) = (c\dot{x}\dot{y} \vee \hat{x}\hat{y}] \cup (d \cdot (\dot{x}\hat{y} \vee \hat{x}\dot{y})]$.

(ii) $X \Rightarrow Y$ *is closed.*

Proof. The consequent part of (i) follows by Lemma 3.7. For the antecedent, we will sketch the proof of the second equality.

Take $\dot{q}$. This, by definition, equals $\bigvee\{a \in A : d \cdot a \in Q\}$. Since $d \cdot \dot{x}\hat{y}$ and $d \cdot \dot{y}\hat{x} = d \cdot \hat{x}\dot{y}$ are both in $X \circ Y$, we get $d \cdot (\dot{x}\hat{y} \vee \dot{y}\hat{x}) \in C(X \circ Y) = Q$. Thus, $\dot{x}\hat{y} \vee \dot{y}\hat{x} \leq \dot{q}$.

To establish the converse inequality, take any $a \in A$ with $d \cdot a \in Q$, and such that $d \cdot a$ is a maximal element of Q. The second closure clause (from Definition 3.3) is taken care of by the maximality requirement, the first and the third cannot produce a genuine (i.e., not majorised in Q by an element of A) maximal $d \cdot a$, so only the fourth clause remains. Let $a = x \vee y$, with $d \cdot x, d \cdot y \in X \circ Y$. Here again there are cases to consider, but, as usual by now, we will take only one as an example. Let $d \cdot x = d \cdot u \cdot w$, with $d \cdot u \in X$, $w \in Y$; and $d \cdot y = d \cdot s \cdot t$, with $d \cdot s \in Y$, $t \in X$. Then, $d \cdot u \leq d \cdot \dot{x}$, $w \leq \hat{y}$, $d \cdot s \leq d \cdot \dot{y}$, $t \leq \hat{x}$. By monotonicity of fusion, it follows that: $d \cdot x \leq d \cdot \dot{x} \cdot \hat{y}$, and $d \cdot y \leq d \cdot \dot{y} \cdot \hat{x}$. By Lemma 3.4, we obtain: $x \leq \dot{x}\hat{y}$, and $y \leq \dot{y}\hat{x}$. It follows that: $a = x \vee y \leq \dot{x}\hat{y} \vee \dot{y}\hat{x}$, precisely as required.

For (ii), let us firstly show that $X \Rightarrow Y$ is downward closed. Let $b \leq a$ and $a \in X \Rightarrow Y$, i.e., $\forall x \in X : a \cdot x \in Y$. By monotonicity of '$\cdot$', and downward closedness of Y, we get: $\forall x \in X : b \cdot x \in Y$, i.e., $b \in X \Rightarrow Y$. Notice also, that $X \Rightarrow Y \supseteq Y$, and thus, $X \Rightarrow Y$ contains $0^{\mathbf{A}}$, for Y does.

For the other two closure clauses, we have—yet again—to proceed case by case. For an instance, let's take: $d \cdot a \in X \Rightarrow Y$ and $d \cdot b \in X \Rightarrow Y$. We have to show that for any $x \in X$, $(d \cdot (a \vee b)) \cdot x \in Y$. We have: $(d \cdot (a \vee b)) \cdot x = d \cdot ((a \vee b) \cdot x)$, by associativity, and here the reasoning splits into two further cases:

(1) if $x \in X|_A$, then as $a, b \in A$ by assumption, we have distributivity of fusion over join, since we remain inside A. Thus, $d \cdot ((a \vee b) \cdot x) = d \cdot (ax \vee bx)$. Now, since $d \cdot a$ and $d \cdot b$ are in $X \Rightarrow Y$, we get: $d \cdot a \cdot x = d \cdot ax \in Y$ and $d \cdot b \cdot x = d \cdot bx \in Y$. By closedness of Y, $d \cdot (ax \vee bx) \in Y$.

(2) if $x \in X \setminus A$, then $x = d_y = d \cdot y$, for some $y \in A$. Thus, $(d \cdot (a \vee b)) \cdot x = (d \cdot (a \vee b)) \cdot (d \cdot y)$. By associativity, this equals $(d \cdot d) \cdot ((a \vee b) \cdot y)$, and we may again employ the distributivity of fusion over join within A; furthermore, $d \cdot d = c$ by definition. Altogether, we obtain: $(d \cdot (a \vee b)) \cdot x = cay \vee cby$. Now, since $d \cdot a \in X \Rightarrow Y$, we have: $d \cdot a \cdot d \cdot y = cay \in Y$. Similarly, for $d \cdot b \cdot d \cdot y = cby$. By closedness of Y, we finally get: $(d \cdot (a \vee b)) \cdot x = cay \vee cby \in Y$.

Definition 3.4. *Let* $\mathbf{C}$ *be the algebra* $\langle C; \vee, \wedge, \cdot, \rightarrow, 1, 0\rangle$, *with the universe* C *being the set of all closed subsets of* B, *and the operations defined as follows:*

- $X \wedge Y = X \cap Y$,
- $X \vee Y = C(X \cup Y)$,
- $X \cdot Y = C(X \circ Y)$,
- $X \rightarrow Y = X \Rightarrow Y$,
- $1^{\mathbf{C}} = B$, $0^{\mathbf{C}} = \{0^{\mathbf{A}}\}$.

The algebra $\mathbf{C}$ that we have just defined will be called the *expansion* of $\mathbf{A}$.

Lemma 3.9. *The expansion* $\mathbf{C}$ *of* $\mathbf{A}$ *is a residuated lattice, and* $\mathbf{A}$ *is a subalgebra of* $\mathbf{C}$.

Proof. That $\mathbf{C}$, with the lattice operations defined as above is indeed a (complete, hence bounded) lattice, follows from the fact that C is a closure operator. Of the other things needed to conclude that $\mathbf{C}$ is a residuated lattice, we only show two: that '$\cdot$' is associative, and that '$\rightarrow$' is indeed the residuation with respect to '$\cdot$'.

For the former, we let Q stand for $Y \cdot Z$, and, employing Lemma 3.8, calculate: $X \cdot (Y \cdot Z) = X \cdot Q = (c\dot{x}\dot{q} \vee \hat{x}\hat{q}] \cup (d \cdot (\dot{x}\hat{q} \vee \hat{x}\dot{q})] = \big(c\dot{x}(\dot{y}\hat{z} \vee \hat{y}\dot{z}) \vee \hat{x}(c\dot{y}\dot{z} \vee \hat{y}\hat{z})\big] \cup \big(d \cdot (\dot{x}(c\dot{y}\dot{z} \vee \hat{y}\hat{z}) \vee \hat{x}(\dot{y}\hat{z} \vee \hat{y}\dot{z}))\big] = \big(c\dot{x}\dot{y}\hat{z} \vee c\dot{x}\hat{y}\dot{z} \vee c\hat{x}\dot{y}\dot{z} \vee \hat{x}\hat{y}\hat{z}\big] \cup \big(d \cdot (c\dot{x}\dot{y}\dot{z} \vee \dot{x}\hat{y}\hat{z} \vee \hat{x}\dot{y}\hat{z} \vee \hat{x}\hat{y}\dot{z})\big]$. This, by the obvious symmetries of the situation (swap z and x and go backwards), equals $(X \cdot Y) \cdot Z$.

For the latter, we need to show that $X \cdot Y \subseteq Z$ iff $X \subseteq Y \Rightarrow Z$. For the left-to-right direction, suppose $x \in X$. Then, take any $y \in Y$. We have $x \cdot y \in X \circ Y \subseteq X \cdot Y \subseteq Z$. Thus, $x \cdot y \in Z$, for any $y \in Y$, which definitionally means $x \in Y \Rightarrow Z$ as needed. For the other direction, observe, that, by the assumption, we have $X \subseteq \{u \in B \mid \forall y \in Y : u \cdot y \in Z\}$, which amounts to: $X \circ Y \subseteq Z$. Since Z is closed, we further get: $X \cdot Y = C(X \circ Y) \subseteq C(Z) = Z$, as needed.

To show that $\mathbf{A} \subseteq \mathbf{C}$, we put $e : A \longrightarrow C$ to be the map defined by $e(a) = (a]$. This is clearly one-one. To prove that it is a homomorphism, we need to show that the following equalities hold:

(i) $(a] \vee (b] = (a \vee^{\mathbf{A}} b]$,

(ii) $(a] \wedge (b] = (a \wedge^{\mathbf{A}} b]$,

(iii) $(a] \rightarrow (b] = (a \rightarrow^{\mathbf{A}} b]$,

(iv) $(a] \cdot (b] = (a \cdot^{\mathbf{A}} b]$,

(v) $(1] = B$, and $(0^{\mathbf{A}}] = \{0^{\mathbf{A}}\}$.

Of these, let us only verify (iii). We have: $(a] \rightarrow (b] = \{z \in B \mid \forall x \leq a : z \cdot x \leq b\}$. Since $(a \rightarrow^{\mathbf{A}} b) \cdot^{\mathbf{A}} a$ equal, by our convention, to $(a \rightarrow^{\mathbf{A}} b)a$ is smaller than b, we get: $\forall x \leq a : (a \rightarrow^{\mathbf{A}} b) \cdot x \leq b$, and hence, $a \rightarrow^{\mathbf{A}} b \in (a] \rightarrow (b]$, i.e., $(a] \rightarrow (b] \supseteq (a \rightarrow^{\mathbf{A}} b]$. For the reverse inclusion, take any $z \in (a] \rightarrow (b]$. Then, $\forall x \leq a : z \cdot x \leq b$. In particular, $z \cdot a \leq b$. If $z \in A$, then $z \leq a \rightarrow^{\mathbf{A}} b$, since $\mathbf{A}$ is a subalgebra of (the partial algebra) $\mathbf{B}$. Suppose $D \ni z = d_y = d \cdot y$, for some $y \in A$. Then, $b \geq z \cdot a = d \cdot y \cdot a$ and, by Lemma 3.4, $b \geq y \cdot a = ya$. Thus, $y \leq a \rightarrow^{\mathbf{A}} b$, hence, $z = d_y \leq a \rightarrow^{\mathbf{A}} b$, as well.

We will sum up the properties of $\mathbf{C}$ in the lemma below. First, however, yet another definition. Let $\mathbf{A}$ be a finite si residuated lattice with the monolithic filter μ.

Definition 3.5. *The filter μ is of depth n iff n is the smallest natural number for which $c^n = c^{n+1} = \star$, where c is the coatom of $\mathbf{A}$.*

Lemma 3.10. *Let $\mathbf{A}$ be a finite si residuated lattice with the monolithic filter μ of depth n, and $\mathbf{C}$ be its expansion. Then, the following hold:*

(i) $\mathbf{A} \subseteq \mathbf{C}$,

(ii) $\mathbf{C}$ *is si,*

(iii) ν, *the monolithic filter of* $\mathbf{C}$, *is of depth* $2n$,

(iv) $\mu = \nu|_{\mathbf{A}}$;

(v) $\mathbf{C}/\nu$ *is isomorphic to* $\mathbf{A}/\mu$.

Proof. Point (i) has already been proved. To prove (ii), it suffices to prove that $(d]$ is the unique coatom of $\mathbf{C}$. It follows from the construction, that d is the unique coatom of $\mathbf{B}$. Thus $(d]$ is the largest proper subset of $\mathbf{B}$. As such, it contains any closed proper subset of B, which proves the claim.

To prove the remaining two claims, observe first, that in the intermediary structure $\mathbf{B}$, we have: $d{\cdot}d = c$, $d \cdot d \cdot c = c \cdot c = c^2$, and, inductively, $d \cdot d \cdot c^k = c^{k+1} = d^{2(k+1)}$. Let n be the depth of μ. Then $c^n = cc^{n-1} = d \cdot d \cdot c^{n-1} < d \cdot c^{n-1} < c^{n-1}$ and $d \cdot d \cdot c^{n-1} = d^{2n}$.

Then, we will prove that for any $0 < k \leq n$ the equality: $(d]^{2k} = (d^{2k}] = (c^k]$, holds. For $k = 1$, we proceed as follows: Take $(d]$ and name it X, for convenience. By Lemma 3.7, X is closed. We will calculate X^2. Observe, that we have: $\dot{x} = 1$, and $\hat{x} = c$. By Lemma 3.8, $X^2 = C(X \circ X)$ equals $(c11 \vee c^2] \cup (d \cdot c] = (c] \cup (d \cdot c]$. As $d \cdot c \leq c$ by definition, we finally get $X^2 = (d]^2 = (d^2] = (c]$. For $k+1$, we get: $(d]^{2k+2} = (d^{2k}] \cdot (d]^2$, which, by inductive hypothesis, equals $(c^k] \cdot (c]$. Now, as previously, let Y, Z, stand, respectively, for $(c^k]$ and $(c]$. These are also closed, by Lemma 3.7. By definition, we get $\dot{y} = \bigvee\{y \in A : d \cdot y \leq c^k\}$, and, since by Lemma 3.4(iv) $d \cdot y \leq c^k$ iff $y \leq c^k$, we obtain: $\dot{y} = c^k$. In similar fashion, we get: $\hat{y} = c^k$, $\dot{z} = c$, and $\hat{z} = c$. Then, $(c^k] \cdot (c] = C(Y \circ Z)$, and this, by Lemma 3.8 and the line above, equals $(cc^k c \vee c^k c] \cup (d \cdot (c^k c \vee c^k c)] = (c^{k+2} \vee c^{k+1}] \cup (d \cdot c^{k+1}] = (c^{k+1}]$, as we claimed. In particular, for $k = n$, we get: $(d]^{2n} = (d^{2n}] = (c^n] = (\star]$, which proves (iii).

Point (iv) follows from (iii) and (i). It remains to show (v). Notice first that $\nu = \mathbf{Cg}^{\mathbf{C}}((d], B)$, and take any closed $X \subseteq B$. By Lemma 3.7, $X = (\hat{x}] \cup (d \cdot \dot{x}]$. Now, we claim that $X \equiv_\nu (\hat{x}]$. This will prove (v), for, since $\hat{x} \in A$ (i.e., $(\hat{x}] \in e(A)$), it will follow that every coset of ν contains an element of A. To prove our claim, it suffices to show that $X \Rightarrow (\hat{x}] \supseteq (\star]$, which in turn, by downward closedness, amounts to $\star \in X \Rightarrow (\hat{x}]$. By definition, $X \Rightarrow (\hat{x}] = \{z \in B \mid \forall x \in X : z{\cdot}x \in (\hat{x}]\}$. By monotonicity of fusion, it is then enough to show that $\star{\cdot}\hat{x} \leq \hat{x}$ and $\star{\cdot}(d{\cdot}\dot{x}) \leq \hat{x}$. The first is trivially true. For the second, we have: $\star{\cdot}(d{\cdot}\dot{x}) = (d{\cdot}\star){\cdot}\dot{x} = \star{\cdot}\dot{x} \leq c{\cdot}\dot{x}$. Writing $\dot{x}$ in full, we obtain: $c{\cdot}\dot{x} = c{\cdot}\bigvee\{z \in A \mid d{\cdot}z \in X\} = \bigvee\{c{\cdot}z \mid z \in A \;\&\; d{\cdot}z \in X\}$. Then, since $c{\cdot}z \leq d{\cdot}z$ and $c \cdot z \in A$, we get that $c \cdot z \leq \hat{x}$, for all $z \in A$ such that $d \cdot z \in X$. Hence, $\bigvee\{c \cdot z \mid z \in A \;\&\; d \cdot z \in X\} \leq \hat{x}$, which proves the claim.

Lemma 3.11. *Let* $\mathbf{A}$ *be a finite si residuated lattice with the monolithic filter* μ *of depth* n*, and* k *be any natural number. Then, there is a finite si residuated lattice* $\mathbf{B}$ *with the monolithic filter* θ*, such that:*

(i) $\mathbf{A} \subseteq \mathbf{B}$,

(ii) θ *is of depth greater than* k,

(iii) $\mu = \theta|_{\mathbf{A}}$,

(iv) $\mathbf{B}/\theta$ *is isomorphic to* $\mathbf{A}/\mu$.

Proof. Let $m \in \omega$ be the smallest such that $2^m n \geq k$. Now, observe that the construction of $\mathbf{C}$ may be iterated. We may "expand" $\mathbf{C}$ in the same way, as we did it with $\mathbf{A}$, and get an algebra $\mathbf{D}$, such that $\mathbf{A}$ is a subalgebra of $\mathbf{D}$, the restriction to $\mathbf{A}$ of the monolithic filter ϕ of $\mathbf{D}$ is precisely μ, the depth of ϕ equals $4n$, and $\mathbf{D}/\phi \cong \mathbf{C}/\nu \cong \mathbf{A}/\mu$ (ν is the monolith of $\mathbf{C}$, as in Lemma 3.10). The m-th iteration of this construction yields the algebra $\mathbf{B}$ satisfying all the claims of the lemma.

Before we finally make use of Lemma 3.2 we need another construction. Let $\mathbf{A}$, $\mathbf{B}$ be finite si residuated lattices. Let c stand for the coatom of $\mathbf{A}$, and q for the coatom of $\mathbf{B}$.

Then, we proceed to define $\mathbf{A} \odot \mathbf{B} = \big\langle \big((A \setminus \{1\}) \times (B \setminus \{1\})\big) \cup \{\langle 1, 1\rangle\}; \wedge, \vee, \cdot, \rightarrow, \langle 1, 1\rangle, \langle 0, 0\rangle \big\rangle$.

We will write a_i, instead of $\langle a, i\rangle$, and 1_1, 0_0 we will further abbreviate to 1, 0, whenever it will not cause confusion. In other words, we view the elements of $A \odot B$ as elements of A indexed by elements of B.

The operations on $A \odot B$ are defined by:

$$a_i \wedge b_j =_{\mathrm{df}} (a \wedge b)_{i \wedge j},$$

$$a_i \vee b_j =_{\mathrm{df}} (a \vee b)_{i \vee j},$$

$$a_i \cdot b_j =_{\mathrm{df}} (a \cdot b)_{i \cdot j},$$

$$a_i \rightarrow b_j =_{\mathrm{df}} \begin{cases} (a \rightarrow b)_{i \rightarrow j}, & \text{if } \ a \not\leq b, \ i \not\leq j; \\ (a \rightarrow b)_q, & \text{if } \ a \not\leq b, \ i \leq j; \\ 1_1, & \text{if } \ a \leq b, \ i \leq j; \\ c_{i \rightarrow j}, & \text{if } \ a \leq b, \ i \not\leq j. \end{cases}$$

Lemma 3.12. $\mathbf{A} \odot \mathbf{B}$ *is an si residuated lattice.*

Proof. First, notice that $A \odot B$, viewed as a subset of $A \times B$ is closed under meet, join, and fusion, and in fact, these have been defined coordinatewise in the clauses above. Thus, $\mathbf{A} \odot \mathbf{B}$ is a lattice ordered commutative monoid, with fusion distributive over join. Moreover, the first and the third clauses in the definition of '$\rightarrow$' are precisely what they would have been in the full product, thus, they indeed define residuation. All that remains is to verify that the second and fourth clauses do so as well.

Let $a \not\leq b$, and $i \leq j$. Then, $(a_i \rightarrow b_j) \cdot a_i = (a \rightarrow b)_q \cdot a_i = ((a \rightarrow b) \cdot a)_{q \cdot i}$. Since $((a \rightarrow b) \cdot a) \leq b$ and $q \cdot i \leq i \leq j$, we obtain $(a_i \rightarrow b_j) \cdot a_i \leq b_j$, as needed. Now, take an element s_k with $s_k \cdot a_i \leq b_j$. Thus, $(s \cdot a)_{k \cdot i} \leq b_j$, and, in particular, $s \cdot a \leq b$. Hence, $s \leq a \rightarrow b$, and since $k \leq q$ is always the case, we obtain $s_k \leq (a \rightarrow b)_q$, as needed.

Let then $a \leq b$ and $i \not\leq j$. We get: $(a_i \rightarrow b_j) \cdot a_i = c_{i \rightarrow j} \cdot a_i = (c \cdot a)_{(i \rightarrow j) \cdot i}$. As $(i \rightarrow j) \cdot i \leq j$ and $c \cdot a \leq a \leq b$, we have $(a_i \rightarrow b_j) \cdot a_i \leq b_j$, as desired. Take a s_k with $s_k \cdot a_i \leq b_j$, i.e., with $(s \cdot a)_{k \cdot i} \leq b_j$. Then, in particular, $k \cdot i \leq j$, and thus, $k \leq i \rightarrow j$. As $s \leq c$ is always the case, we have $s_k \leq c_{i \rightarrow j} = a_i \rightarrow b_j$, as desired.

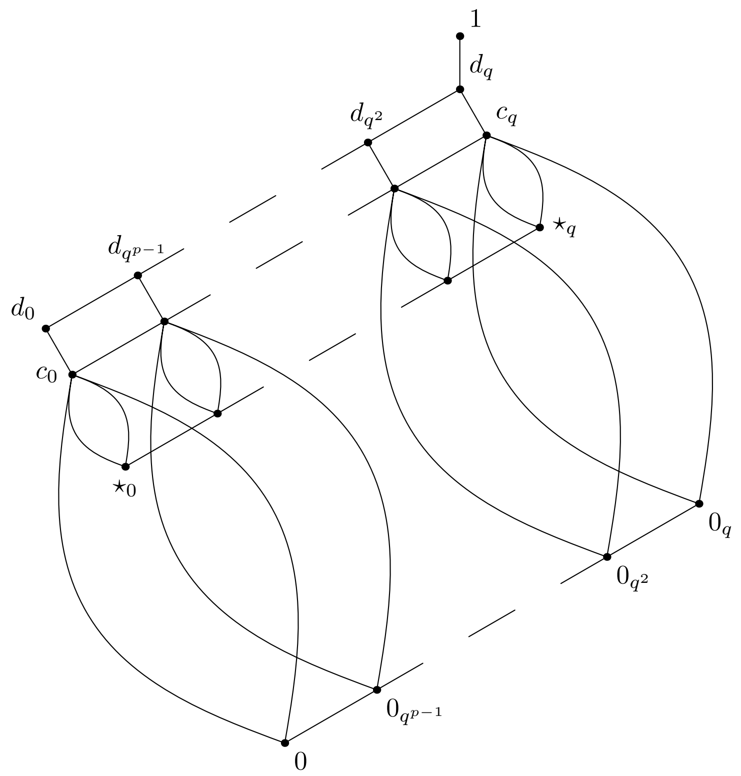
Being si is obvious, just take the congruence filter generated by c_q.

Now, let $\mathbf{A}$, μ, $\mathbf{B}$, $n \leq k \in \omega$ be as in Lemma 3.11, and let $m \geq k$ be the depth of ν. Consider $\mathbf{B} \odot \mathbf{Ł}_{p+1}$, where $\mathbf{Ł}_{p+1}$ is the simple Łukasiewicz algebra with $p+1$ elements, for the first prime number p greater or equal to $|B|$. Let also c, d, and q stand for the unique coatoms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{Ł}_{p+1}$, respectively.

Lemma 3.13. *For any nontrivial congruence θ on $\mathbf{B} \odot \mathbf{Ł}_{p+1}$, $\mathbf{B} \odot \mathbf{Ł}_{p+1}/\theta$ is isomorphic to a proper homomorphic image of $\mathbf{A}$.*

Proof. As $p \geq m$, we get $(d_q)^p = d^p_{q^p} = \star_0$. Thus, every nontrivial θ has $1\theta\star_0$. Since $a_q \rightarrow a_0 = d_{q \rightarrow 0} \geq \star_{q \rightarrow 0} \geq \star_0$, we also have: $a_i \theta a_j$, for any $i, j \in Ł_{p+1} \setminus \{1\}$. Consider the relation $\sim$ on $B \odot Ł_{p+1} \times B \odot Ł_{p+1}$, defined as $\{(x,y) : x = a_i, y = a_j, \text{for some } a \in B \text{ and some } i,j \in Ł_{p+1}\}$. So defined, $\sim$ is a congruence on the lattice reduct of $\mathbf{B}\odot\mathbf{Ł}_{p+1}$. Clearly, $\theta \geq \sim$. From the construction, it is readily seen that $\langle B\odot Ł_{p+1}/{\sim}; \wedge, \vee\rangle$ and $\langle B; \wedge, \vee\rangle$ are isomorphic as lattices (see the picture below). Now the conclusion will follow by Lemma 3.11(iv), if we show that θ collapses the ("projection" of the) monolith ν of $\mathbf{B}$. This happens, if $\theta \ni (\star_0, 1)$, and precisely this has been established at the beginning of the proof.

The picture below shows what $\mathbf{B} \odot \mathbf{Ł}_{p+1}$ looks like. Notice, that the number of iterations required for an element d_i, with any index i, to descend below $\star_q$ depends only on the "distance" between the line of d's and the line of $\star$'s; in other words, on the depth of ν.



Lemma 3.14. *If* $\mathbf{A}$ *is not isomorphic to* $\mathbf{2}$*, then* $\mathbf{A} \notin SH(\mathbf{B} \odot \mathbf{Ł}_{p+1})$*; therefore,* $\mathbf{A} \notin \mathcal{V}(\mathbf{B} \odot \mathbf{Ł}_{p+1})$

Proof. By the previous fact it follows that $|B \odot Ł_{p+1}/\theta| < |A|$, and thus $\mathbf{A}$ cannot be a subalgebra of any proper homomorphic image of $\mathbf{B} \odot \mathbf{Ł}_{p+1}$.

Suppose that there is a map $e : A \leftrightarrow B \odot Ł_{p+1}$ which embeds $\mathbf{A}$ into $\mathbf{B} \odot \mathbf{Ł}_{p+1}$. Notice that if, for any $a \neq b$, $e(a) = v_i$ and $e(b) = u_i$, then, assuming as we can that $a \to b \neq 1$, we obtain $e(a \to b) = v_i \to u_i = (v \to u)_q$. Let w stand for $v \to u$. Consider the chain $w_q, (w_q)^2, \ldots, (w_q)^p$. By definition, its elements are respectively equal to $w_q, w^2_{q^2}, \ldots, w^p_{q^p}$. As $q, q^2, \ldots, q^p$ are all distinct by the construction, the chain above has p distinct elements. This contradicts the assumptions about $\mathbf{A}$. Thus, each (image of an) element of A must belong to a different "layer" in $B \odot Ł_{p+1}$ (see the picture above). Since $\mathbf{A}$ is not isomorphic to $\mathbf{2}$, A has at least three elements. Thus, for some $a \in A$ we must have $e(a) = a_i$ with $i \neq 0, 1$. However, since in $\mathbf{Ł}_{p+1}$ each non-zero and non-unit element generates the whole algebra, we obtain that a_i generates p distinct elements in $B \odot L$. This leads to the same contradiction again.

The rest follows by Jónsson Lemma from the fact that $\mathcal{R}$ has CEP, $\mathbf{A}$ is si, and $\mathbf{B} \odot \mathbf{Ł}_{p+1}$ is finite.

Lemma 3.15. *There is a valuation v such that $\mathbf{B} \odot \mathbf{Ł}_{p+1} \not\models_v Y_{\mathbf{A}}^{(k)} = 1$, for any $k < p$.*

Proof. Take the valuation v with $v(x_1) = 1$ and $v(x_a) = a_q$, if $a \neq 1$. For $\diamond \in \{\vee, \wedge, \rightarrow\}$, we have $v(x_{a\diamond b} \leftrightarrow x_a \diamond x_b) = (a \diamond b)_q \leftrightarrow (a_q \diamond b_q) = (a \diamond b)_q \leftrightarrow (a \diamond b)_q = 1$. In the case of fusion, we have: $v(x_{a\cdot b} \leftrightarrow x_a \cdot x_b) = (a \cdot b)_q \leftrightarrow (a_q \cdot b_q) = (a \cdot b)_q \leftrightarrow (a \cdot b)_{q\cdot q} = (a \cdot b)_q \rightarrow (a \cdot b)_{q\cdot q} = d_{q\rightarrow(q\cdot q)} = d_q$. The case of negation splits into three subcases. For $a \notin \{0, 1\}$, we obtain: $v(x_{\neg a} \leftrightarrow \neg x_a) = v(x_{a\rightarrow 0} \leftrightarrow (x_a \rightarrow 0_0)) = (a \rightarrow 0)_q \leftrightarrow (a_q \rightarrow 0_0) = (a \rightarrow 0)_q \leftrightarrow (a \rightarrow 0)_{q\rightarrow 0} = (a \rightarrow 0)_q \rightarrow (a \rightarrow 0)_{q\rightarrow 0} = d_{q\rightarrow(q\rightarrow 0)} = d_{\neg q^2} = d_{q^{p-2}}$. For $a = 1$, we get: $v(x_0 \leftrightarrow \neg x_1) = 0_q \leftrightarrow 0_0 = d_{\neg q} = d_{q^{p-1}}$. For $a = 0$, we get: $v(x_1 \leftrightarrow \neg x_0) = 1 \leftrightarrow \neg 0_q = 1 \rightarrow (0_q \rightarrow 0_0) = 1_1 \rightarrow d_{q\rightarrow 0} = (1 \rightarrow d)_{1\rightarrow\neg q} = d_{\neg q} = d_{q^{p-1}}$.

Altogether, $v(\Delta_{\mathbf{A}}) = c_{q^{p-1}}$. Now, since the depth of the monolith of $\mathbf{B}$ is m, the properties of $\mathbf{B} \odot \mathbf{Ł}_{p+1}$ guarantee that $(d_{q^{p-1}})^k \nleq \star_q$ for any $k < m$. To see that, recall that the lattice operations and the fusion in $\mathbf{B} \odot \mathbf{Ł}_{p+1}$ behave exactly as they would have in the direct product $\mathbf{B} \times \mathbf{Ł}_{p+1}$. Thus, for any k with $1 < k < m$, we have $(d_{q^{p-1}})^k = d^k_{(q^{p-1})^k} = d^k_0$. Moreover, $d^k_0 = d^k_q \wedge d_0$, and thus, as fusion and meet are computed coordinatewise, we obtain: $d^k_0 = d^k_q \wedge d_0 \leq \star_q$ iff $d^k \wedge d \leq \star$ in $\mathbf{B}$ and $q \wedge 0 \leq q$ in $\mathbf{Ł}_{p+1}$. By construction of $\mathbf{B}$, however, we have $d^k \wedge d = d^k > d^m = \star$ in $\mathbf{B}$; hence $d^k_0 \nleq \star_q$, which is the desired conclusion. For $k = 1$ the argument is analogous, with the index q^{p-1} in place of 0.

Thus, we have $v(Y_{\mathbf{A}}^{(k)}) = v(\Delta^k_{\mathbf{A}} \rightarrow x_\star) = (d_{q^{p-1}})^k \rightarrow \star_q$. Hence, $\mathbf{B} \odot \mathbf{Ł}_{p+1} \not\models_v Y_{\mathbf{A}}^{(k)} = 1$.

Conclusion and some remarks

We are now ready to prove the main result of the chapter.

Theorem 3.1. *The only algebra that splits $L^v(\mathcal{R})$ is the two-element boolean algebra* **2**.

Proof. That **2** splits $L^v(\mathcal{R})$ follows by Proposition 3.3. Let $\mathbf{A}$ be a finite si residuated lattice different from **2**. Take any $k \in \omega$. Let $\mathbf{B}$ be the algebra whose existence is guaranteed by Lemma 3.11. Then, by Lemma 3.15, $\mathbf{B} \odot \mathbf{Ł}_{p+1}$ falsifies $Y_{\mathbf{A}}^{(k)} = 1$. By Lemma 3.14, $\mathbf{A} \notin \mathcal{V}(\mathbf{B} \odot \mathbf{Ł}_{p+1})$. Together, these constitute precisely the condition (ii) of Lemma 3.2, by which the theorem follows.

Naturally, one would like to know how far the above result would go. Certainly, for any variety $\mathcal{V}$ *containing* $\mathcal{R}$ it dramatically limits chances for splittings in $L^v(\mathcal{V})$, although, of course, it does not prove that there will not be splitting algebras outside $\mathcal{R}$. However, it does not seem unreasonable to hope that the constructions used in the proof, or some modifications thereof will work. One way of generalising our concept of a residuated lattice is to give up commutativity of multiplication, and possibly adding another binary function, as residuation then splits naturally into left and right-residuation. This is in fact, the concept of residuated lattices found e.g., in [2]. Let $\mathcal{R}^-$ stand for the variety of such algebras.

Question 3.1. *Are there any non-boolean splitting algebras in $\mathcal{R}^-$?*

In this case, we are bold enough to conjecture that the answer is no. On the other hand, for subvarieties of $\mathcal{R}$ the outlook is much gloomier. Take the distributive subvariety. Then, our first construction does not have much chance for survival, as completions are notorious for destroying distributivity. Take the linear one. Whereas the first construction is not immediately doomed, the second dies a sudden death (never produces anything linear, except in trivial cases). Thus, the following seems to us a interesting project.

Question 3.2. *Investigate splittings in subvarieties of $\mathcal{R}$.*

It looks particularly worthwhile for subvarieties associated naturally to some well-known logics. Linear residuated lattices and basic algebras come to mind. A first step in this project could be to answer the next:

Question 3.3. *Which subvarieties of $\mathcal{R}$ without EDPC have non-boolean splitting algebras?*

4. ALMOST MINIMAL VARIETIES

As we saw in the previous chapter, the structure of the whole $L^v(\mathcal{R})$ is quite complicated. Thus, it seems reasonable to look at very small subvarieties of $\mathcal{R}$ first. This type of approach is known among universal algebraists by the jargon phrase "description of the bottom of the lattice of subvarieties". Clearly, the rule of thumb here is: the fewer small subvarieties the better chances for a good description. Such a description naturally begins at the level of atoms (i.e., *minimal* varieties, to be defined shortly), and then—if possible—proceeds to covers of atoms, and further on. Again, the results of the chapter will be on the negative side, for we show that although $L^v(\mathcal{V})$ for any nontrivial subvariety of $\mathcal{R}$ has precisely one atom, even fairly small subvarieties of $\mathcal{R}$ can contain continuum covers (which we call *almost minimal*) of the atom. All the results were obtained jointly with M. Ueda (cf. [53], [34]).

Almost minimality

A variety $\mathcal{V}$ is called *minimal* iff $\mathcal{V}$ has only one, trivial, proper subvariety. Clearly, a minimal variety must be generated by a single si algebra, equally clearly, the converse is not true. No general characterisation of these si algebras which in fact do generate minimal varieties exists at present, although quite a lot is known in this direction (cf. e.g., Szendrei [51]). Since every nontrivial residuated lattice has the two-element Boolean algebra as a subalgebra, there is only one minimal variety of residuated lattices, namely the variety $\mathcal{BA}$ of Boolean algebras.

We call a variety of residuated lattices *almost minimal* iff it has only one nontrivial proper subvariety. In other words, a variety $\mathcal{V} \subseteq \mathcal{R}$ is almost minimal iff $\mathcal{V}$ covers $\mathcal{BA}$ in the lattice of subvarieties of $\mathcal{R}$.

Let $\mathbf{A}$ be an si residuated lattice. We say that $\mathbf{A}$ is *stiff* iff (1) $\mathbf{A}$ has exactly one proper nontrivial quotient isomorphic to the two element BA, (2) $\mathbf{A}$ has no proper subalgebras other than the two element BA.

Lemma 4.1. *Let $\mathbf{A}$ be a stiff residuated lattice, with the monolith μ. The following hold in $\mathbf{A}$:*

(i) *there is precisely one $b \in A \setminus \{0,1\}$, with $b^2 = b$;*

(ii) *for any $a \in A$, $a \in F_\mu$ iff $a \geq b$;*

(iii) *for any $a \in A$, $a \in F_\mu$ or $\neg a \in F_\mu$;*

(iv) **Con A** *is a three-element chain.*

Moreover, if for some positive integer n, $\mathbf{A} \in \mathcal{E}_n$, all the above can be expressed by first-order formulas.

Proof. Since (i) is already a first-order formula, the claim follows by noticing that for every $c \in A \setminus \{0,1\}$, with $c^2 = c$ has $\{x \in A : x \geq c\}$ as a congruence filter, and thus there is precisely one such, since $\mathbf{A}$ is stiff. For (ii) note that by (i) b is a definable constant; moreover, by EDPC for $\mathcal{E}_n$, we can express '$a \in F_\mu$'

by a first-order formula. If (iii) did not hold, then $\mathbf{A}/\mu$ would fail to be a BA. Finally, (iv) holds in $\mathbf{A}$ by definition, and is equivalent to the following: for any $a \in A$, if $a \notin F_\mu$, then the congruence generated by $(a, 1)$ is the full congruence. This, again by EDPC, can be expressed as a first-order formula.

Lemma 4.2. *For any equation $\sigma \approx \tau$, we have: $\sigma \approx \tau$ is true in the two-element BA iff $\sigma \to \tau \in F_\mu$ and $\tau \to \sigma \in F_\mu$ iff $\mathbf{A} \models \phi(\sigma, \tau)$, where $\phi(\sigma, \tau)$ is a certain first-order formula definable from σ and τ. In particular, the fact that $\mathbf{A}/\mu$ is isomorphic to the two-element BA is expressible by a first-order formula.*

Proof. The first equivalence follows from the definition of stiff residuated lattice and the fact that $a \to b \in F_\mu$ iff $a/\mu \leq b/\mu \in \mathbf{A}/\mu$. For the second it suffices to recall that '$a \in F_\mu$' is first-order definable. For the last statement note that $\mathbf{A}/\mu$'s being isomorphic to the two-element BA could be restated as the conjunction of '$\mathbf{A}/\mu$ has precisely two elements' and 'all equations that hold in BAs hold in $\mathbf{A}/\mu$'. The former is expressible by a first-order sentence, regardless of the type (provided we have identity in the type). The latter is further equivalent to 'all axioms of BAs hold in $\mathbf{A}/\mu$' and that, by finite axiomatisability of BAs, is a finite conjunction of first-order formulas.

Lemma 4.3. *Let $\mathbf{A} \in \mathcal{E}_n$ be stiff. All the claims from Lemmas 4.1 and 4.2 hold for any ultrapower $\mathbf{A}^I/U$ of $\mathbf{A}$.*

Proof. As $\mathbf{A}$ and $\mathbf{A}^I/U$ are elementarily equivalent, $\mathbf{A}^I/U$ satisfies all the properties of $\mathbf{A}$ expressible by first-order formulas. Since $\mathbf{A} \in \mathcal{E}_n$, all the claims from Lemma 4.1 are such. Thus, the congruence lattice of $\mathbf{A}^I/U$ is a three-element chain, and therefore $\mathbf{A}^I/U$ is si. Let ν stand for its monolith. Then, '$a \in F_\nu$' is first-order definable on $\mathbf{A}^I/U$, which suffices to conclude that the claims from Lemma 4.2 also hold.

Lemma 4.4. *Let $\mathbf{A}$ be a stiff algebra in $\mathcal{E}_n$, for some $n \in \omega$, and let $\mathbf{C}$ be an si algebra in $\mathcal{V}(\mathbf{A})$ nonisomorphic to the two-element BA. Then, $\mathbf{A}$ is a subalgebra of $\mathbf{C}$.*

Proof. By Jónsson Lemma, $\mathbf{C}$ belongs to $HSP_U(\mathbf{A})$. By CEP, $\mathbf{C} \in SHP_U(\mathbf{A})$. Thus, there is an ultrapower $\mathbf{A}^I/U$, with I any set of indices and U an ultrafilter on I, and a congruence ψ on $\mathbf{A}^I/U$, such that $\mathbf{C} \subseteq (\mathbf{A}^I/U)/\psi$. By Lemma 4.3, $\mathbf{A}^I/U$ is si, and the quotient of $\mathbf{A}^I/U$ by its monolith ν is isomorphic to the two-element BA. Thus, our congruence ψ can only be trivial. It follows that $\mathbf{C} \subseteq \mathbf{A}^I/U$. By Lemma 4.1 again, there is precisely one element $b \in A^I/U \setminus \{0, 1\}$, with $b^2 = b$, and, moreover, any $a \in A^I/U \setminus \{0, 1\}$ has either $a^n = b$ or $(\neg a)^n = b$. Since the canonical embedding of $\mathbf{A}$ into $\mathbf{A}^I/U$ sends idempotent elements to idempotent elements, we have that b generates $\mathbf{A}$ as a subalgebra of $\mathbf{A}^I/U$. Thus, $\mathbf{A} \subseteq \mathbf{C}$.

Lemma 4.5. *Each stiff member of $\mathcal{E}_n$, for any $n \in \omega$, generates an almost minimal variety. If $\mathbf{A} \in \mathcal{E}_n$ and $\mathbf{B} \in \mathcal{E}_n$ are nonisomorphic stiff algebras, then $\mathcal{V}(\mathbf{A}) \neq \mathcal{V}(\mathbf{B})$.*

Proof. Let $\mathbf{A}$ be a stiff algebra in $\mathcal{E}_n$, and $\mathbf{C}$ be any non-Boolean si algebra in $\mathcal{V}(\mathbf{A})$. Clearly, $\mathcal{V}(\mathbf{A}) \supseteq \mathcal{V}(\mathbf{C})$. By Lemma 4.4, $\mathbf{A} \subseteq \mathbf{C}$. Thus, $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{C})$, which proves the first statement.

For the second, it suffices to prove that $\mathbf{A} \notin \mathcal{V}(\mathbf{B})$. Suppose the contrary, then by Lemma 4.4, $\mathbf{B} \subseteq \mathbf{A}$. Since $\mathbf{A}$ is stiff, it has only two subalgebras: the two-element BA and itself. Since the two-element BA is not stiff, we get that $\mathbf{B} \cong \mathbf{A}$. This contradicts the assumption.

As there are only countably many finite stiff residuated lattices, our purpose requires constructing infinite ones. This, however, is much less difficult than it may seem, at least for varieties with EDPC, since "stiffness", being a first-order property, carries over to ultraproducts. Assume that we work within a subvariety $\mathcal{V}$ of $\mathcal{E}_k$, for some positive integer k. Suppose that for each $n \in \omega$, we have a finite stiff residuated lattice $\mathbf{A}_n$, with $|A_n| > n$. Recall from Lemma 4.1 that each of these has a unique idempotent element different from 0 and 1. We will reserve the symbol $\star$ for it. Let now $\mathbf{A} = \prod_{n\in\omega} \mathbf{A}_n/U$, be an ultraproduct of all the $\mathbf{A}_n$ by a free ultrafilter U on ω. Then, let $\mathbf{A}^\star$ be the subalgebra of $\mathbf{A}$ generated by $\star = \langle \star_n : n \in \omega\rangle/U$.

Lemma 4.6. *The algebra $\mathbf{A}^\star$ is an infinite stiff residuated lattice.*

Proof. By EDPC, the ultraproduct $\mathbf{A}$ is si, and its congruence lattice is a three-element chain. Thus, the same goes for $\mathbf{A}^\star$, and to show that it is stiff we only need to prove that every $a \in A^\star \setminus \{1, 0\}$ generates $\mathbf{A}^\star$. Clearly, it suffices to show that a generates $\star$. If $a \in F_\star$ (with $F_\star$ being the monolithic congruence filter), then $a^n = \star$. Otherwise, $\neg a \in F_\star$, and thus $(\neg a)^n = \star$.

Now to show that $\mathbf{A}^\star$ is infinite, note first that the elements generated by $\star$ in $\mathbf{A}$ are congruence classes of elements generated in $\prod_{n\in\omega} \mathbf{A}_n$ by $\langle \star_n : n \in \omega\rangle$. As usual we view the generation process as follows: for any $n \in \omega$, let $G^0_n = \{\star_n\}$, and $G^{k+1}_n = \{F(g) : g \in G^k_n, F - \text{a basic operation}\}$. Suppose $\mathbf{A}^\star$ is finite. Then, by properties of ultraproducts, any element x generated by $\langle \star_n : n \in \omega\rangle$ at every $k+1$-th stage of generation "later" than some fixed m, belongs to G^k_i at almost all coordinates i. This implies that $G^{k+1}_i = G^k_i$, for almost all i. Since the type is finite, $|G^k_i|$ is bounded by a finite number that does not depend on i (its very rough estimate is $\underbrace{4(4(\cdots(4}_{k-1}\cdot 3^{\overbrace{2)^2\ldots)^2}^{k-1}}$, for $k \geq 2$). It follows that infinitely many $\mathbf{A}_i$ have sizes bounded by the same finite number, which contradicts the assumption that $|A_n| > n$ for each n.

Constructions at work

It follows from Lemma 4.6 that one way of constructing uncountably many almost minimal varieties of residuated lattices can go as follows. Working within $\mathcal{E}_k$, for some positive integer k, construct uncountably many families of finite stiff algebras, each family containing algebras of unbounded size. Say, for each $S \subseteq \omega$, construct a family $(\mathbf{A}_{S,n})_{n\in\omega}$, with $|\mathbf{A}_{S,n}| \geq n$, for each $n \in \omega$. Then, for each S, $\mathbf{A}^\star_S$, is stiff. The task will be completed if we manage to show that $\mathbf{A}^\star_S$ is nonisomorphic to $\mathbf{A}^\star_T$, whenever $S \neq T$.

We will present three such constructions, the first with some detail, the other two rather briefly.

The classical case

Let S be any subset of ω. For any positive integer K, define C^S_K be the disjoint union of the sets: $A = \{a_0, \ldots, a_K\}$, $B = \{b_{-1}, b_0, \ldots, b_K\}$, $M = \{m_{-1}, m_0, \ldots, m_K\}$, $N = \{n_0, \ldots, n_K\}$, and $E = \{\mathbf{0}, \mathbf{1}\}$. C^S_K is partially ordered (see Figure 1) by the transitive reflexive closure of the relation '$\triangleright$' defined below:

- $\mathbf{1} \triangleright a_0$, $n_0 \triangleright \mathbf{0}$;

- $a_i \triangleright a_j$ iff $i < j$, $b_i \triangleright b_j$ iff $i < j$;
- $m_i \triangleright m_j$ iff $i > j$, $n_i \triangleright n_j$ iff $i > j$;
- $a_K \triangleright b_{-1}$, $m_{-1} \triangleright n_K$;
- $b_0 \triangleright m_K$, $b_K \triangleright m_0$.

Define multiplication on C_K^S, putting:

- $\mathbf{1} \cdot x = x$, $\mathbf{0} \cdot x = \mathbf{0}$, $x \cdot y = y \cdot x$, for all $x, y \in C_K^S$;
- $a_i \cdot a_j = \begin{cases} b_{\max\{i,j\}-1}, & \text{if } i \neq j,\ \max\{i,j\} \in S, \text{ or} \\ & \quad \min\{i,j\} = 0, \\ b_{\max\{i,j\}}, & \text{if } \min\{i,j\} > 0,\ \max\{i,j\} \notin S, \text{ or} \\ & \quad i = j > 0; \end{cases}$
- $a_i \cdot b_j = b_K$;
- $a_i \cdot n_j = \begin{cases} \mathbf{0}, & \text{if } i \geq j, \\ n_0, & \text{if } i < j; \end{cases}$
- $a_i \cdot m_j = \begin{cases} n_0, & \text{if } i > j, \\ n_1 & \text{if } i = j \in \{0,1\}, \text{ or} \\ & \quad i = j > 1,\ j \notin S, \\ n_j, & \text{if } i = j > 1,\ j \in S, \text{ or} \\ & \quad 0 < i < j,\ j \notin S, \\ n_{j+1}, & \text{if } 0 = i < j < K, \text{ or} \\ & \quad 0 < i < j < K,\ j \in S, \\ m_{-1}, & \text{if } 0 = i < j = K, \text{ or} \\ & \quad 0 < i < j = K,\ K \notin S; \end{cases}$
- $b_i \cdot b_j = b_K$;
- $b_i \cdot m_j = \begin{cases} \mathbf{0}, & \text{if } i \geq j, \\ n_0, & \text{if } i < j; \end{cases}$
- $b_i \cdot n_j = \mathbf{0}$;
- $m_i \cdot m_j = \begin{cases} \mathbf{0}, & \text{if } \min\{i,j\} \leq 0, \\ n_0, & \text{otherwise}; \end{cases}$
- $m_i \cdot n_j = \mathbf{0}$;
- $n_i \cdot n_j = \mathbf{0}$.

Define residuation on C_K^S, putting:

- $x \to y = \bigvee\{z \in C_K^S : z \cdot x \leq y\}$.

Finally, let $\mathbf{C}_K^S$ be the algebra $\langle C_K^S; \vee, \wedge, \cdot, \to, \mathbf{0}, \mathbf{1}\rangle$.

Lemma 4.7. *The algebra $\mathbf{C}_K^S$ is a residuated lattice.*

Proof. We have to prove two things: that multiplication above, commutative by definition, is also associative, and that multiplication distributes over join, i.e., $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$ holds. For associativity we have to consider several cases, of which only one is not straightforward and is dealt with below:

$$(a_i \cdot a_j) \cdot m_k = \begin{cases} b_{j-1} \cdot m_k, & \text{if } i = 0, \\ b_{i-1} \cdot m_k, & \text{if } j = 0, \\ b_{j-1} \cdot m_k, & \text{if } 0 < i < j,\ j \in S, \\ b_i \cdot m_k, & \text{if } 0 < i = j, \\ b_j \cdot m_k, & \text{if } 0 < i < j,\ j \notin S, \\ b_i \cdot m_k, & \text{if } 0 < i > j,\ i \notin S; \end{cases}$$

which yields further:

$$(a_i \cdot a_j) \cdot m_k = \begin{cases} \mathbf{0}, & \text{if } j-1 \geq k,\ i = 0, \text{ or} \\ & \quad i-1 \geq k,\ j = 0, \text{ or} \\ & \quad j-1 \geq k,\ 0 < i < j,\ j \in S, \text{ or} \\ & \quad i-1 \geq k,\ 0 < j < i,\ i \in S, \text{ or} \\ & \quad i \geq k,\ 0 < i = j, \text{ or} \\ & \quad j \geq k,\ 0 < i < j,\ j \notin S, \text{ or} \\ & \quad i \geq k,\ 0 < j < i,\ i \notin S; \\ n_0, & \text{if } j-1 < k,\ i = 0, \text{ or} \\ & \quad i-1 < k,\ j = 0, \text{ or} \\ & \quad j-1 < k,\ 0 < i < j,\ j \in S, \text{ or} \\ & \quad i-1 < k,\ 0 < j > i,\ i \in S, \text{ or} \\ & \quad i < k,\ 0 < i = j, \text{ or} \\ & \quad j < k,\ 0 < i < j,\ j \notin S, \text{ or} \\ & \quad i < k,\ 0 < j < i,\ i \notin S. \end{cases}$$

Then, changing the bracketing, we get:

$$a_i \cdot (a_j \cdot m_k) = \begin{cases} a_i \cdot n_0, & \text{if } j > k, \\ a_i \cdot n_1 & \text{if } j = k \in \{0,1\}, \text{ or} \\ & \quad j = k > 1,\ k \notin S, \\ a_i \cdot n_k, & \text{if } j = k > 1,\ k \in S, \text{ or} \\ & \quad 0 < j < k,\ k \notin S, \\ a_i \cdot n_{k+1}, & \text{if } 0 = j < k < K, \text{ or} \\ & \quad 0 < j < k < K,\ k \in S, \text{ or} \\ a_i \cdot m_{-1}, & \text{if } 0 = j < k = K, \text{ or} \\ & \quad 0 < j < k = K,\ K \in S. \end{cases}$$

which finally yields:

$$a_i \cdot (a_j \cdot m_k) = \begin{cases} \mathbf{0}, & \text{if } j > k, \text{ or} \\ & \quad i \geq 1,\ j = k \in \{0,1\}, \text{ or} \\ & \quad i \geq 1,\ j = k > 1,\ k \notin S, \text{ or} \\ & \quad i \geq k,\ j = k > 1,\ k \in S, \text{ or} \\ & \quad i \geq k,\ 0 < j < k,\ k \notin S, \text{ or} \\ & \quad i \geq k+1,\ 0 = j < k < K, \text{ or} \\ & \quad i \geq k+1,\ 0 < j < k < K,\ k \in S; \\ n_0, & \text{if } i = 0,\ j = k \in \{0,1\}, \text{ or} \\ & \quad i = 0,\ j = k > 1,\ k \notin S, \text{ or} \\ & \quad i < k,\ j = k > 1,\ k \in S, \text{ or} \\ & \quad i < k,\ 0 < j < k,\ k \notin S, \text{ or} \\ & \quad i < k+1,\ 0 = j < k < K, \text{ or} \\ & \quad i < k+1,\ 0 < j < k < K,\ k \in S, \text{ or} \\ & \quad 0 = j < k = K, \text{ or} \\ & \quad 0 < j < k = K,\ K \in S. \end{cases}$$

The proof now reduces to a series of tedious paper-and-pencil verifications confirming that the cases with **0** and the cases with n_0 are indeed mutually exclusive.

For distributivity of multiplication over join, we proceed in two steps. First, we show that multiplication is monotone, i.e., $x \leq y$ implies $z \cdot x \leq z \cdot y$. Out of a number of cases only two deserve attention:

The first is $z = a_i$, $x = a_j$, $y = a_k$. As $a_j \leq a_k$, we have $j \geq k$. We may assume $j > k$. Then, if $i \geq j > k$, both $a_i \cdot a_j$ and $a_i \cdot a_k$ are equal to either of b_i, b_{i-1}. The only dubious case arises when $a_i \cdot a_j = b_{i-1}$. This, however, can happen only if $i \in S$, and then $a_i \cdot a_k = b_{i-1}$ as well. If $j > i \geq k$, we get $a_i \cdot a_j \in \{b_{j-1}, b_j\}$ and $a_i \cdot a_k \in \{b_{i-1}, b_i\}$. Since $j > i$, we have $b_{j-1} \leq b_i$, which establishes the claim for all subcases. Finally, if $j > k > i$, we have $a_i \cdot a_j \in \{b_{j-1}, b_j\}$ and $a_i \cdot a_k \in \{b_{k-1}, b_k\}$. As $j > k$, the previous reasoning applies.

The second is $z = a_i$, $x = m_j$, $y = m_k$. Here we have $j \leq k$, and we may also assume $j < k$. Then, if $i \leq j$, we have $a_i \cdot m_j \in \{n_1, n_j, n_{j+1}\}$, and only the cases (1) $a_i \cdot m_j = n_j$ and (2) $a_i \cdot m_j = n_{j+1}$ are not straightforward. Case (1) splits into two: (1a) $i = j > 1$ and $j \in S$, in which case $k > i > 1$, and thus $a_i{\cdot}m_k \in \{n_j, n_{j+1}, m_{-1}\}$, proving the claim; (1b) $0 < i < j$ and $j \notin S$, in which case $a_i{\cdot}m_k \in \{n_k, n_{k+1}, m_{-1}\}$, proving the claim as well, for $j < k$, by the assumption. Case (2) also splits into two: (2a) if $0 = i < j < K$, then $a_i \cdot m_k\{n_{k+1}, m_{-1}\}$, and since $j < k$, the claim is proved; (2b) if $0 < i < j < K$ and $j \in S$, then $a_i \cdot m_k\{n_k, n_{k+1}, m_{-1}\}$; now the assumption guarantees that $j + 1 \leq k$, therefore the claim holds here as well.

Having established monotonicity, we can approach the proof of distributivity. Consider $x(y \vee z)$. If y and z are compatible, say $y \geq z$, then $x(y \vee z) = xy = xy \vee xz$, by monotonicity. Suppose y and z are incompatible. We may take $y = b_i$ ($i \in \{1, \ldots, K\}$) and $z = m_j$ ($j \in \{1, \ldots, K\}$). Then, $x(y \vee z) = x(b_i \vee m_j) = xb_0$, and $xy \vee xz = xb_i \vee xm_j$. If $x = \mathbf{1}$, or $x = \mathbf{0}$, then the desired equality obviously holds. Assume $x \notin \{\mathbf{1}, \mathbf{0}\}$. Then, if $x \geq b_K$, we have: $x(y \vee z) = xb_0 = b_K$, and $xy \vee xz = xb_i \vee xm_j = b_K$, since $xm_j \leq m_0$. Thus, the equality holds here, too. If $x \leq m_K$, then two cases should be distinguished: (1) $x \leq m_0$, in which case $x(y \vee z) = xb_0 = \mathbf{0}$, and $xy \vee xz = xb_i \vee xm_j = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$; (2) $x = m_k$, with $k \in \{1, \ldots, K\}$, in which case $x(y \vee z) = m_k b_0 = n_0$, and $xy \vee xz = m_k b_i \vee m_k m_j = n_0$, as well. This finishes the whole proof.

Lemma 4.8. *The residuated lattice $\mathbf{C}_K^S$ is stiff. Moreover, it belongs to $\mathcal{E}_3$ and satisfies $\neg\neg x = x$.*

Proof. It is clear from the construction that $\mathbf{C}_K^S$ satisfies $x^4 = x^3$. It is also clear, that the only nontrivial and non-full congruence on $\mathbf{C}_K^S$ is the one associated with the filter $F = \{x \in C_K^S : x \geq b_K\}$. The quotient of this congruence is the two-element Boolean algebra. We also have: $\neg a_i = n_i$, $\neg b_i = m_i$, $\neg n_i = a_i$, and $\neg m_i = b_i$; we leave out the detailed calculations. This shows that $\neg\neg x = x$ holds. Moreover, for any $x \in C_K^S \setminus \{\mathbf{1}, \mathbf{0}\}$, we have either $x^3 = b_K$, or $(\neg x)^3 = b_K$. Thus, to prove stiffness and finish the whole proof, it suffices to show that $\mathbf{C}_K^S$ is generated by b_K. Take $b_K \vee \neg b_K = b_K \vee m_K = b_0$. This generates a_0, since $b_0 \to b_K = a_0$. Further, $a_0 a_0 = b_{-1}$. Then, $a_0 \to b_0 = a_1$, and $a_1^2 = b_1$. Suppose $a_0, \ldots, a_n$, b_{-1}, $b_0, \ldots, b_n$ have been generated. Then, $a_0 \to b_n = a_{n+1}$, and $a_{n+1}^2 = b_{n+1}$. This shows that all the elements in A and B get generated. Then the sets N and M are generated by negation, and this finishes the generation process. Observe that this process is independent from the set S, in the sense that, had S been chosen differently, the operations involved in the generation would still yield precisely the same results.

Consider now the family $(\mathbf{C}_K^S)_{K \in \omega^+}$, where $\omega^+ = \omega \setminus \{0\}$. Take any free ultrafilter U on ω and let $\mathbf{C}_S$ stand for the ultraproduct $\prod_{K \in \omega^+} \mathbf{C}_K^S / U$. Further, let $\mathbf{C}_S^\star$ be the subalgebra of $\mathbf{C}_S$ generated by $\star$. Clearly, the assumptions of Lemma 4.6 apply to our construction. Thus, $\mathbf{C}^\star_S$ is an infinite stiff residuated lattice, and as such it generates an almost minimal variety.

Lemma 4.9. *For any S, T subsets of $\omega \setminus \{0,1\}$, we have: if $S \neq T$, then $\mathbf{C}^\star_S$ is not isomorphic to $\mathbf{C}^\star_T$.*

Proof. Before we embark on the proof, let us dwell for a while on what $\mathbf{C}_S$ and $\mathbf{C}_T$ look like. We will refer to certain elements of these ultraproducts by the names of the elements of factor algebras, for instance, a_i in the appropriate context, will stand for $\langle e(n) : n \in \omega^+\rangle/U$, where $e(n) = a_i(n)$, if a_i exists in C^S_n, or is arbitrary otherwise. Notice, that $\star$ is always unambiguous, being a definable constant.

Now, suppose S, T are subsets of $\omega \setminus \{0,1\}$ and $S \neq T$, yet $\mathbf{C}^\star_S$ and $\mathbf{C}^\star_T$ are isomorphic. We can therefore identify the lattices underlying $\mathbf{C}^\star_S$ and $\mathbf{C}^\star_T$. Now, to obtain the desired contradiction, we look at multiplication and residuation induced on these lattices.

Let i be the smallest number such that $i \in S$ but $i \notin T$; we can always assume such a number exists, if not we just swap S and T. Consider the elements a_1, a_i; note that we use these names unambiguously, because, as the generation process from the previous proof does not depend on the choice of S and T, we may use a_1 as shorthand for $((\star \vee \neg\star) \to \star) \to (\star \vee \neg\star)$, and a_i as a shorthand for something similar, only much longer. Now, consider $a_1 \cdot^S a_i$. Since $0 < 1 < i \in S$, the first clause in the definition of '$\cdot$' applies at almost all coordinates, yielding in the ultraproduct: $a_1 \cdot^S a_i = b_{i-1}$. Then, for $a_1 \cdot^T a_i$, we have $0 < 1 < i \notin T$ and the second clause applies, yielding: $a_1 \cdot^S a_i = b_i$. However, b_i and b_{i-1} are different in the ultraproduct, hence, $a_1 \cdot^S a_i$ and $a_1 \cdot^T a_i$ produce different results, contradicting the assumption that $\mathbf{C}^\star_S$ and $\mathbf{C}^\star_T$ are isomorphic.

Theorem 4.1. *There are $2^{\aleph_0}$ almost minimal subvarieties of $\mathcal{E}_3 \cap \mathcal{C}$.*

For contrast, we quote the following, from [53] (see also [45]).

Theorem 4.2. *There are 2 almost minimal subvarieties of $\mathcal{E}_2 \cap \mathcal{C}$.*

The linear case

For any S with $0 \in S \subseteq \omega$. For any positive integer K define L^S_K to be the disjoint union of the sets: $B = \{b_0, \dots, b_{K+1}\}$, $A = \{a_0, a_1, a_2\}$, $N = \{n_0, \dots, n_{K+1}\}$, $C = \{\mathbf{0}, \mathbf{1}\}$, and $D = \{d_s : s < K,\ s \in S\} \cup \{e\}$. L^S_K is totally ordered (see Figure 1) by the transitive reflexive closure of the relation '$\triangleright$' defined below:

- $\mathbf{1} \triangleright b_0 \triangleright b_{K+1} \triangleright a_0 \triangleright a_1 \triangleright a_2 \triangleright n_{K+1} \triangleright d_0 \triangleright e \triangleright \mathbf{0}$;
- $b_i \triangleright b_{i+1}$, $n_{i+1} \triangleright n_i$, for all $i \in \{0, \dots, K\}$;
- $d_r \triangleright d_s$ iff $r < s$ in the natural ordering of S.

For any $i \in \omega$, let $\lfloor i \rfloor$ stand for the largest $s \in S$ with $s \leq i$. Such an s always exists, for $0 \in S$. With this notation at hand, we define multiplication on C^S_K, putting:

- $\mathbf{1} \cdot x = x$, $\mathbf{0} \cdot x = \mathbf{0}$, $x \cdot y = y \cdot x$, for all $x, y \in L^S_K$;

- $b_i \cdot b_j = \begin{cases} a_0, & \text{if } \min\{i,j\} = 0, \\ a_1, & \text{if } 0 < i,\ 0 < j < K, \text{ or} \\ & \quad 0 < j,\ 0 < i < K, \text{ or} \\ & \quad i = j = K, \\ a_2 & \text{if } i \in \{K, K+1\},\ j = K+1, \text{ or} \\ & \quad j \in \{K, K+1\},\ i = K+1; \end{cases}$
- $b_i \cdot a_j = a_2$
- $b_i \cdot n_j = \begin{cases} d_{\lfloor i \rfloor}, & \text{if } j \leq i, \\ e, & \text{if } j = i+1, \\ \mathbf{0}, & \text{if } j > i+1; \end{cases}$
- $x \cdot y = \mathbf{0}$, in all other cases.

Residuation is defined as previously:

- $x \to y = \bigvee\{z \in C_K^S : z \cdot x \leq y\}$.

Then, let $\mathbf{L}_K^S$ be the algebra $\langle L_K^S; \vee, \wedge, \cdot, \to, \mathbf{0}, \mathbf{1}\rangle$.

Lemma 4.10. *The algebra $\mathbf{L}_K^S$ is a residuated lattice.*

Proof. Left for the reader.

Lemma 4.11. *The residuated lattice $\mathbf{L}_K^S$ is stiff. Moreover, it belongs to $\mathcal{E}_3$ and satisfies $x \to y \vee y \to x = 1$.*

Proof. The only thing that is not immediately seen from the construction is that $\mathbf{L}_K^S$ has no subalgebras apart from the two-element Boolean algebra and itself. To show this it suffices to prove that the element a_2 generates $\mathbf{L}_K^S$. We have: $\neg a_2 = n_{K+1}$ and $\neg n_{K+1} = b_{K+1}$. Then, $b_{K+1} \to a_2 = b_K$. Further, $b_K \cdot n_{K+1} = e$, and $\neg b_K = n_K$, $n_K \to e = b_{K-1}$. Then, by backward induction on i, $\neg b_i = n_i$, $n_i \to e = b_{i-1}$ and thus we have generated all the elements, except a_0, a_1 and all the elements d_s, for $s \in S \cap \{0, \ldots, K-1\}$. To get these, we may, for instance, employ: $b_0^2 = a_0$, $b_1^2 = a_1$, and finally $b_{K+1} \cdot n_s = d_s$, for all suitable s.

As in the previous case, let $\mathbf{L}_S$ be the ultraproduct $\prod_{K \in \omega^+} \mathbf{L}_K^S/U$ by a free ultrafilter U on ω and $\mathbf{L}_S^\star$ its subalgebra generated by $\star$. Again, Lemma 4.6 applies, thus $\mathbf{L}_S^\star$ is an infinite stiff algebra.

Lemma 4.12. *Let S, T be subsets of ω, each containing 0. If $S \neq T$, then $\mathbf{L}_S^\star$ is not isomorphic to $\mathbf{L}_T^\star$.*

Proof. We adopt all the conventions from Lemma 4.10. It will also be convenient (and consistent with the way ultraproducts work in nonstandard models for arithmetic) to view K, used in $b_K = \langle b_K(n) : n \in \omega\rangle/U$, as an infinitely large natural number, so that $K > n$, for any $n \in \omega$. Suppose i is the smallest number with $i \in S \setminus T$. Observe that in both $\mathbf{L}_S^\star$ and $\mathbf{L}_T^\star$ we have a finite increasing sequence d_0, $d_{\lfloor 1 \rfloor}, \ldots, d_{\lfloor i-1 \rfloor}$, with k distinct terms ($k \leq i$). Then, $b_i \cdot^S n_{K+1} = d_{\lfloor i \rfloor} = d_i \neq d_{\lfloor i-1 \rfloor} = b_{i-1} \cdot^S n_{K+1}$, since $i \in S$; but $b_i \cdot^T n_{K+1} = d_{\lfloor i \rfloor} = d_{\lfloor i-1 \rfloor} = b_{i-1} \cdot^T n_{K+1}$, as $i \notin T$.

Theorem 4.3. *There are $2^{\aleph_0}$ almost minimal subvarieties of $\mathcal{E}_3 \cap \mathcal{L}$.*

Again, the following is proved in [53].

Theorem 4.4. *There are* 6 *almost minimal subvarieties of* $\mathcal{E}_2 \cap \mathcal{L}$.

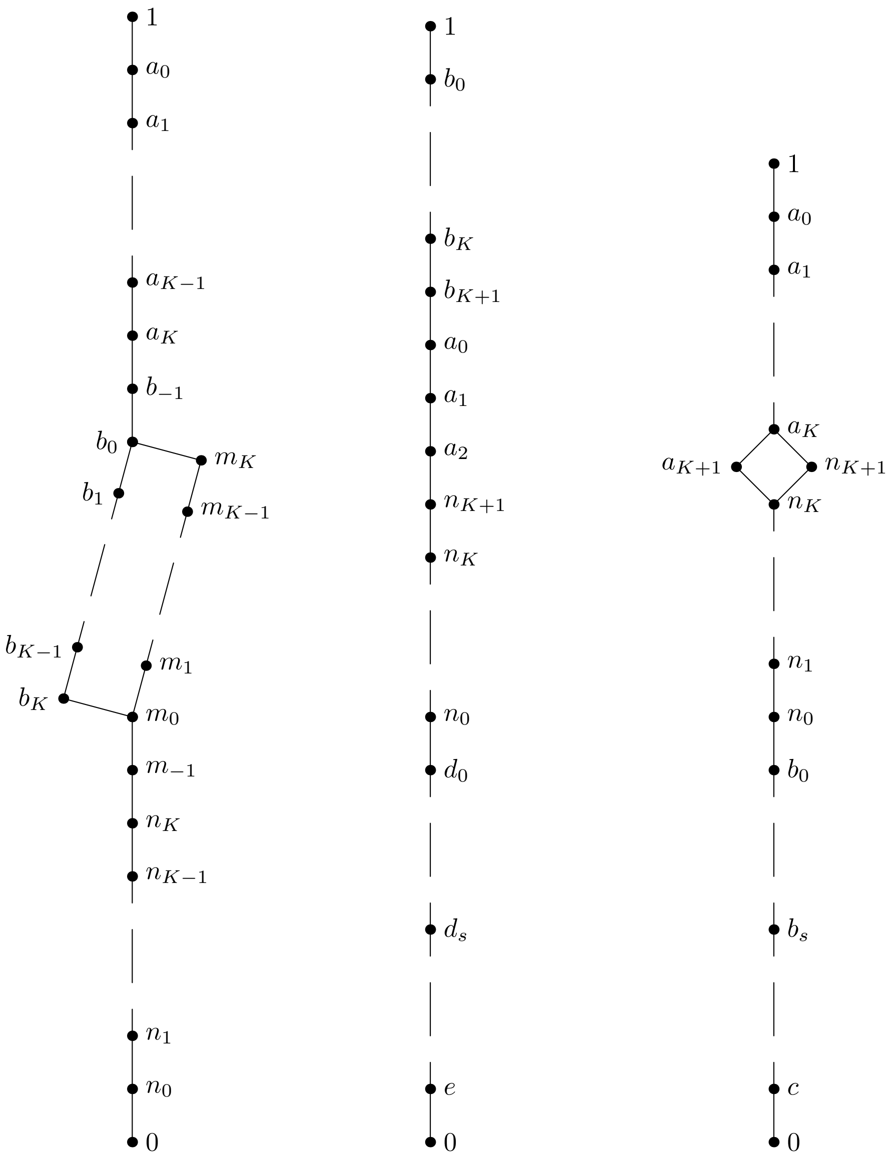


Figure 1. Lattice structures of our algebras. From the left: $\mathbf{C}_K^S$, $\mathbf{L}_K^S$, and $\mathbf{D}_K^S$.

The distributive case

We finish off with yet one more construction, this time descending as low as $\mathcal{E}_2$ in the subvarieties of $\mathcal{R}$. However, as all the proofs proceed exactly as in previous cases, none will be presented.

Let S be a subset of ω with with $0 \in S$. For any positive integer K, let D_K^S be the disjoint union of the sets: $A = \{a_0, \ldots, a_{K+1}\}$, $N = \{n_0, \ldots, n_{K+1}\}$, $C = \{\mathbf{0}, \mathbf{1}\}$, $B = \{b_s : s < K,\ s \in S\} \cup \{c\}$. D_K^S is partially ordered (see Figure 1) by the transitive reflexive closure of the relation '$\triangleright$' defined below:

- $\mathbf{1} \triangleright a_0$, $n_0 \triangleright b_0$, $c \triangleright \mathbf{0}$;
- $x \triangleright c$, for all $x \neq \mathbf{0}$;
- $a_i \triangleright a_j$, $n_j \triangleright n_i$ iff $i < j$ $(i, j \in \{0, \ldots, K+1\})$;
- $a_K \triangleright n_{K+1}$, $a_{K+1} \triangleright n_K$;
- $b_r \triangleright b_s$ iff $r < s$ in the natural ordering of S.

For any $i \in \omega$, let $\lfloor i \rfloor$ stand, as previously, for the largest $s \in S$ with $s \leq i$. Define multiplication on D^S_K, putting:

- $\mathbf{1} \cdot x = x$, $\mathbf{0} \cdot x = \mathbf{0}$, $x \cdot y = y \cdot x$, for all $x, y \in D^S_K$;
- $a_i \cdot a_j = a_{K+1}$;
- $a_i \cdot n_j = \begin{cases} b_{\lfloor i \rfloor}, & \text{if } j > i+1, \\ c, & \text{if } j = i+1, \\ \mathbf{0}, & \text{if } j \leq i; \end{cases}$
- $x \cdot y = \mathbf{0}$, in all other cases.

Lemma 4.13. *The algebra $\mathbf{D}^S_K$ is a stiff residuated lattice. Moreover, it belongs to $\mathcal{E}_2$ and is distributive as a lattice.*

Lemma 4.14. *Let S, T be subsets of ω, each containing 0. If $S \neq T$, then $\mathbf{L}^\star_S$ is not isomorphic to $\mathbf{L}^\star_T$.*

Theorem 4.5. *There are $2^{\aleph_0}$ almost minimal subvarieties of $\mathcal{E}_2 \cap \mathcal{D}$.*

Questions and remarks

Almost minimal varieties of residuated lattices that are generated by infinite si algebras have one curious property that may be worth spelling out. Let $\mathcal{V} \subset \mathcal{R}$ be such a variety. Then, for any $\mathbf{A} \in \mathcal{V}$ is finite iff $\mathbf{A}$ is a Boolean algebra. This has several rather peculiar consequences, one of which is that all such varieties have decidable first-order theory of their finite algebras, but, clearly, $2^{\aleph_0}$ of them have undecidable equational theories.

Let $\mathbf{A}_S$ be any of the infinite algebras discussed above, with S being the subset of ω employed in its construction. In the context the following question suggests itself:

Question 4.1. *Is it true that the equational theory of $\mathbf{A}_S$ is decidable iff S is recursive?*

Also the question of possible finite axiomatisability of equational theory of $\mathbf{A}_S$ may be of interest. There exist at least one almost minimal variety of residuated lattices, generated by an infinite algebra, whose equational theory is indeed finitely axiomatisable, however it does not arise from our constructions. It is the variety of *product algebras* (see, e.g., [16]).

Our results may at first sight suggest that any subvariety of $\mathcal{R}$ that contains infinitely many almost minimal varieties, contains $2^{\aleph_0}$ of them. This would be too far-fetched. The variety $\mathcal{WA}$ of Wajsberg algebras provides a counterexample (cf. [33]). Yet, $\mathcal{WA}$ does not contain infinitely many stiff algebras (in fact, it contains none). Moreover, $\mathcal{WA}$ is not contained in $\mathcal{E}_n$, for any n. In view of this it seems sensible to ask:

Question 4.2. *Is it true that if $\mathcal{V} \subseteq \mathcal{E}_n$, for some positive integer n, contains infinitely many stiff algebras, then it contains $2^{\aleph_0}$ almost minimal subvarieties.*

If the answer to the above question is in fact negative, then the varieties $\mathcal{V}(\{\mathbf{A}_K^S : K \in \omega\})$, with $\mathbf{A} \in \{\mathbf{C}, \mathbf{L}, \mathbf{D}\}$ are candidates for a counterexample.

Our method of constructing almost minimal varieties, although fairly general, depends crucially on two factors: the existence of special infinite families of finite residuated lattices, and the fact that these algebras are stiff (in particular, on the fact that $\star \neq 0$). If we focus on almost minimal varieties generated by simple algebras the second factor disappears completely, for in these $\star = 0$; as for the first, not much is known. Therefore, we ask:

Question 4.3. *Are there rather many, or rather few almost minimal varieties of residuated lattices generated by simple algebras?*

At present we only know that there are at least countably many of these in $\mathcal{R}$; again, varieties generated by finite simple Łukasiewicz algebras with $p+1$ elements, for a prime p are examples. However, even that much is not known about subvarieties of $\mathcal{E}_n$, for $n > 2$. For $n = 1, 2$, the answers are respectively 0 (folklore) and 1 (Ueda).

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