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Bolzano's Concept of Consequence

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Plainly, to identify a speech as an argument and to understand its premises and conclusion is not the same as knowing what argument is intended. What is missing?

Bernard Bolzano defines the concept of consequence thus:

> Propositions M, N, O,... *follow* from propositions A, B, C, D,... with respect to variable parts i, j,... if every class of ideas whose substitution for i, j,... makes each of A, B, C, D,... true also makes all of M, N, O,... true.[1]

The i, j... are constants tagged for substitution; I shall call them *variands*. We do not often state them explicitly, but resort to hints like "It follows by modus ponens..." or "He found himself on the horns of a dilemma...." Usually we rely on convention and context. But this leaves many arguments irremediably opaque. Bolzano gives the following example:

> Since all humans have an irresistible yearning for enduring existence, and since even the most virtuous must be miserable in the thought that some day they will cease to exist, we justly expect from God's infinite benevolence that he will not annihilate us in death (WL § 164, No. 2).

If this is intended as an argument, then, following Bolzano, some of its constants must be meant as variands; but it is wholly unclear which. If no variands were intended, then this would merely be a string of assertions.

It is tempting to think that if a putative argument is not so muddled as to resist all analysis, then a transcription into modern symbolic notation, the textbook exercise "Put this argument in symbolic form," would identify the variands. Bolzano would deny this. He holds, rather, that to state a consequence unambiguously requires an explicit listing of variands. Since this is not done even in rigorously formal contexts, ambiguities can occur even there.

Consider $A \supset B, B \supset A \models A \supset A$. Is this meant to turn on the conclusion's being a previously established theorem, or is it a syllogism? The ambiguity is removed if variands are appointed, e.g., by drawing boxes around them, thus:

(1) $\boxed{A \supset B}, \boxed{B \supset A} \models \boxed{A} \supset \boxed{A}$

(2) $\boxed{A} \supset \boxed{B}, \boxed{B} \supset \boxed{A} \models \boxed{A} \supset \boxed{A}$

In this dispensation, these are distinct arguments, both valid. (1) shows that the premises do not matter, since boxed items, even if complex, may be (uniformly) replaced by arbitrary sentences, even atoms. (2), on the other hand, is a syllogism. For the classical logician, the distinction does not matter. He can think of the argument(s) as derived from the schema $p, q \models r \supset r$, which yields only valid arguments. But Bolzano's concept of form is different. Here the word, and not the variable, stands at the beginning, and the form is identified

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with the set of arguments generated from a given argument by variation, i.e., substitution on variands.[2] A form, then, is not a schema. Arguments are not "obtained" from forms by the substitution of constants for variables, and it is not, strictly speaking, correct to say that forms are valid or invalid, though we may say, by extension, that forms are valid if they contain only valid arguments, and invalid if this is not so.

The conception of consequence here adumbrated has two features that should recommend it to logicians who are concerned not with the development of formal systems, but with the analysis of informally stated arguments and the identification of fallacies.

The first of these is that arguments of invalid form are invalid. In the classical view, this is not the case, as Gerald Massey has pointed out with clarity and vigor.[3] For example, the schema "Affirming the Consequent" has as instances certain valid arguments, e.g., $A \supset A$, $A \models A$. More importantly, every classical syllogism is an instance of the invalid form $p, q \models r$. Since this sort of thing sometimes occurs, one is never justified, so the argument goes, in judging an argument fallacious just because it is an instance of an invalid form, particularly given the apparently unfinished state of logic.

In Bolzano's view, the evaluation of any argument must begin with the identification of variands. If their variation generates an invalid form, the argument is invalid; if not, not. It is of course possible to make mistakes in this, just as sentences can be misunderstood. It is a cultural, and perhaps even a human, failing that we do not usually indicate the variands explicitly. But these are problems of communication. Plainly, it is often possible, and sometimes important, to identify formal fallacies. It therefore seems that in this respect Bolzano's account of consequence is superior to the classical.

A second positive feature of Bolzano's conception is that it gives a promising account of enthymemes. Although he concentrates on arguments in which all indexical elements are variands (this being the proper province of logic, cf. WL §223), his definition does not exclude cases in which only some of them are. We readily identify 'Socrates' as the variand in 'Socrates was a man, therefore Socrates was mortal'. That is, we understand this argument as implicitly claiming that every substitution on 'Socrates' that makes the premise true also makes the conclusion true. If we had to construct a device for computing the "missing premise" (which we intuitively take to be 'All men are mortal'), we would have it state that fact. It would, that is, form the universal closure on the variand, over the conditional consisting of premise and conclusion, and *voilà*, the missing premise results. This procedure works for all syllogistic enthymemes, and is only slightly more complex when no singular terms are involved. No principle of charity or other proviso is needed. I venture the guess that some such computation is going on even in our own minds when, with a speed that must compel wonder, we determine what all the world takes to be the missing premise in such a case.

I must correct a point I made in earlier papers,[4] to the effect that the variands in an enthymeme are the items shared by premise and conclusion. It is an accident that this works for syllogisms. More complex enthymemes require a different treatment. Consider "If I buy cheap tickets, my girl friend will be angry. I buy either cheap or expensive tickets. Therefore either my girl friend will be angry, or I will be broke." In a flash we see that a suitable missing premise is 'If I buy expensive tickets, then I will be broke'. This dictates that the variands be 'I buy cheap tickets' and 'My girl friend will be angry'. But to determine the variands in this way is to put the cart before the horse. There must be a procedure that finds them first and computes the missing premise from that information. But I do not know, at this time, what it is. And matters would be even more complex if unneeded premises were present.

Bolzano's construal of consequence also sheds some light on the "deductivism" controversy. If deductivism means that only deductive arguments are good arguments, then Bolzano assuredly is an anti-deductivist. It is just wrong, in his view, to cast enthymemes, which are often good arguments, into the same darkness as the grossest *non sequiturs*, and a principled formal treatment of them is here suggested. On the other hand, he could conceivably be called a deductivist because he does not allow purely material arguments, that is, arguments without variands. I myself find the notion of such an argument unintelligible, and will not pursue the matter.

Let me now state some formal features of Bolzano consequence, and some other matters, without proof. Since these consequences are triads, some of the points could not be stated in the conventional rendition of arguments.

Bolzano defines the concepts of compatibility and analyticity as follows: a set of sentences Γ is *analytic* with respect to a set of variands υ if every substitution on the elements of υ makes all its elements true, and *compatible* with respect to *itupsilon* if some substitution on υ makes all of them true. Allow the list of variands to contain items that do not occur in the argument, or the set of sentences; that is, allow this list to contain "idle" elements. This makes it possible to specify, for example, that the list should contain all and only atomic sentences. (In this case the consequence will be said to be *classical.*)

Indicating the set of variands above the turnstile, abbreviating 'Γ is compatible (analytic) with respect to υ' as Comp (Γ; υ) and Anal (Γ;υ), and writing 'υ, A' for '$\upsilon \cup \{A\}$', the following hold:[5]

I. If A is a sentence idle in υ, A, then $\Gamma \overset{\upsilon}{\models} \Delta$ iff $\Gamma \overset{\upsilon, A}{\models} \Delta$, Comp ($\Gamma$;$\upsilon$) iff Comp ($\Gamma$;$\upsilon, A$), and Anal ($\Gamma$;$\upsilon$) iff Anal ($\Gamma$;$\upsilon, A$).

II. If $\Gamma \overset{\upsilon}{\models} \Delta$ and $\Delta \overset{\upsilon}{\models} \Theta$, then $\Gamma \overset{\upsilon}{\models} \Theta$ (WL §155, No. 24).

III. If $\Gamma \overset{\upsilon, \omega}{\models} \Delta$ with ω idle, and $\Delta \overset{\upsilon, \omega}{\models} \Theta$ with υ idle, then $\Gamma \overset{\upsilon, \omega}{\models} \Theta$ (WL §155, No. 23).

IV. If $\Gamma \overset{\upsilon}{\models} \Delta$ and $\Gamma \overset{\upsilon}{\models} \Theta$, then $\Gamma \overset{\upsilon}{\models} \Delta, \Theta$ (WL §155, No. 22.).

With this in hand, let us next look at C. I. Lewis's "Independent Proof" for *ex absurdo quodlibet*. From the viewpoint of Bolzano consequence it appears as a sophism, at least in the form in which it is usually presented.

Lewis's argument can be rendered thus:

(1)	$A \,\&\, -A$	Premise
(2)	A	from 1 by simplification
(3)	$-A$	from 1 by simplification
(4)	$A \vee B$	from 2 by weakening
(5)	B	from 3 and 4 by disjunctive syllogism

A rigorous rendition in Bolzano style invokes the theorems I, II, and IV above. This is left as an exercise.

The first and second steps are said to be based on the law that from any conjunction either conjunct may be inferred. From our point of view, this can only mean that the variands in the premise are 'A' and '$-A$'. Only then (i.e., if 'A' and '$-A$' are varied independently of each other: $\boxed{A,}\,\boxed{-A} \models \boxed{A}$) can the form (i.e., the set generated from the premise) contain *all* conjunctions. And, presumably, *this* set is envisaged in the talk about 'any conjunction.' But, if this is so, then the disjunctive syllogism cannot be executed, for '$-A$' does not now function as the negation of 'A'. The variands that allow the argument to go through are 'A' and 'B'. But, if *they* are chosen, then the law that sanctions the first two steps is not "From a conjunction infer either conjunct" but, rather, "From the conjunction of a sentence and its denial infer either" – hardly a law of much dignity, and one that we shall presently dispute.

The sophism, from our point of view, lies in a peculiar kind of equivocation. We have seen that arguments, like sentences, can be ambiguous. Now Lewis's "Independent Proof" must be read one way (with variands 'A', 'B') to make it go through, and another way (with variands 'A', '$-A$', 'B') to make good on the claim that the laws in question are, as Lewis put it, "unavoidable consequences of indispensable laws of inference."[6] But on this construal the argument fails.

Bolzano imposed a further condition. He required that the premises of any consequence by *compatible* with respect to the set of variands υ of the consequence. He did not forbid the conclusion(s) of a consequence to be analytic with respect to υ (cf., e.g., WL §155 No. 12). This asymmetry is awkward and leads, e.g., to the rejection of transposition. I shall deviate here from Bolzano and explore a consequence relation that requires the premises to be compatible and the conclusion(s) to be nonanalytic with respect to the variands. Let us call the consequence relation so defined "Bolzano Consequence in the Narrow sense" (BCN), symbolized by 'N='.

BCN comes close to an entailment relation suggested by T. J. Smiley, who takes it that a true entailment is "a substitution instance of a tautological implication whose components are neither contradictory nor tautological."[7] Alan Ross Anderson and Nuel Belnap have proved that if an entailment relation satisfies this condition, then it is *relevant* in the sense of sharing a subsentence.[8] The same holds for BCN:

V. If $\Gamma \,\mathrm{N}\overset{\upsilon}{=} \Delta$, then υ contains a variand common to both Γ and Δ.

Suppose that it does not. Then υ can be represented as the disjoint set υ', υ'', where υ' contains only variands in Γ, and υ'' only variands in Δ. But Comp(Γ;υ), and, since υ'' is here idle in υ, it follows by (I) that Comp($\Gamma;\upsilon'$). Analogously, since not Anal($\Gamma;\upsilon$), and υ' is here idle, it follows that not Anal($\Delta;\upsilon''$). There are then substitutions on υ, i.e., on υ' in Γ and on υ'' in Δ, which make the former true and the latter false. Hence, contrary to our assumption, $\Gamma N \overset{\upsilon}{=} \Delta$ fails. So premises and conclusion share a subsentence.

It is plain that Lewis's "Independent Proof" fails in BCN, for the first two steps must now be construed with 'A' and '$-A$' as variands. It is thus not the disjunctive syllogism that fails, but the chain: we are constrained (so as to be able to begin) to construe the argument in this way, and when we come to the disjunctive syllogism we are stopped. Deductive explosions are thus made impossible.

The following can be said in support of BCN: we must expect a reasoner to know what his argument is. That is, he must know his variands. If he does not, he is in fact not reasoning, but asserting strings of sentences. Is it then too much to ask that the premises be compatible, and the conclusions not analytic, with respect to these same variands? Are we not simply asking that he should not *consciously* employ inconsistent premises? Indeed, by requesting this we are not asking even that he should assure that his premises are taken from a consistent subregion of a possibly inconsistent set of propositions. We merely require that his premises may be deemed consistent by someone who does not fully understand them, but knows the variands. BCN allows inconsistencies in a body of beliefs, which may creep into premises of arguments, and yet allows the logician the cherished freedom of saying important things about matters he does not understand and of which he does not know whether they are true.

Good things come at a price. In this case the price is *case argument*. In discrediting the disjunctive syllogism, Anderson and Belnap point (by implicaiton) to the following: the premise of $(A \vee B)\ \&\ -A \models B$ is equivalent to $(A\ \&\ -A) \vee (A\ \&\ B)$. By case argument we obtain $A - A \models B$. Thus it seems that the acceptance of the disjunctive syllogism leads at once to *ex absurdo quodlibet*, a result to be avoided. But case argument does not hold in BCN. Instead we have the rule:

$$\frac{\Gamma \vee \Delta\ N \overset{\upsilon}{=} \Theta}{\Gamma\ N \overset{\upsilon}{=} \Theta \text{ and } \Theta\ N \overset{\upsilon}{=} \Theta \text{provided that comp } (\Gamma;\upsilon) \text{ and comp}(\Delta;\upsilon)}$$

In other words, reasoner must recheck the compatibility of his premises with respect to variands after each application of case argument, and remove those which don't satisfy the compatibility requirement. For, plainly, if $A\ \&\ -A N \overset{A}{=} ?$ cannot be the first step of an argument, it can also not be the *n*th step.

The classical notion of consequence relies on the principle, always unstated, that arguments are derived from schemata, as if a reasoner, like the demiurge, always gazed upon forms while creating his product. We can now see that, in BCN, this can be framed as an antitheorem. Suppose a consequence had a molecular variand. Replace it by its immediate subsentence(s). Call the new argument "a more fine-grained associate" of the first. The anti-theorem then is "If an argument is valid, then all its more fine-grained associates also are." If this is preferred to the compatibility requirement, then Bolzano consequence collapses into classical. But, conceivably, BCN is a workable alternative, though further exploration may disclose monstrous complications that diminish its initial attractiveness.

Notes

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1 *Wissenschaftslehre* (Sulzbach, 1837), §155, no. 2, vol. II, pp. 199 ff. Translated as *Theory of Science*, R. George, ed. (Oxford: Blackwell, 1972), p. 209. Henceforth WL.

2 The use of the term 'form' as synonym for 'species' or 'set' is justified by a whimsical reference to Cicero: "*Quod enim utroque verbo* [i.e., '*forma*' and '*species*'] *idem significatur.*" Cicero, *Topica* 30. Bolzano held that logic is a formal science in *this* sense. Cf. WL §81, note 1.

3 "The Fallacy behind Fallacies," in P. A. French, T. E. Vehling, Jr., and H. K. Wettstein, eds., *The Foundations of Analytic Philosophy* (Minneapolis: Minnesota UP, 1981), pp. 499 ff.

4 "Enthymematic Consequence," *American Philosophical Quarterly*, ix, 1 (January 1972): 113–16, and

"Bolzano's Consequence, Relevance, and Enthymemes," *Journal of Philosophical Logic*, XII, 3 (August 1983): 299–318, p. 315.

5 Bolzano has many more theorems than are here given, and he gives slightly different versions than we have here. Also, to do propositional logic in this system, one needs a rule of precedence: the longest elements of v should be "boxed" first, then the next longest, down to the atoms, if any.

6 C. I. Lewis and C. H. Langford, *Symbolic Logic* (New York: Dover, 1959), p. 512.

7 "Entailment and Deducibility," *Proceedings of the Aristotelian Society*, LIX, (1959): 233–254.

8 *Entailment* (Princeton, N.J.: University Press, 1975), pp. 215–20.