

Derivations in K .

- Any propositional tautologies
- $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ and $\Box\alpha \leftrightarrow \sim\Diamond\sim\alpha$.
- Uniform substitution + Modus Ponens + Modal Generalization

$$(MP) \quad \frac{\alpha, \alpha \rightarrow \beta}{\beta} \qquad (MG) \quad \frac{\alpha}{\Box\alpha}$$

in $S4$: extra axioms: $\Diamond\Diamond\alpha \rightarrow \Diamond\alpha$ and $\alpha \rightarrow \Diamond\alpha$ (dually: $\Box\alpha \rightarrow \Box\Box\alpha$ and $\Box\alpha \rightarrow \alpha$).

1. $\vdash_K (\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$

1. $p \rightarrow (q \rightarrow (p \wedge q))$
2. $\Box(p \rightarrow (q \rightarrow (p \wedge q)))$
3. $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
4. $\Box(p \rightarrow (q \rightarrow (p \wedge q))) \rightarrow (\Box p \rightarrow \Box(q \rightarrow p \wedge q))$
5. $\Box p \rightarrow \Box(q \rightarrow p \wedge q)$
6. $\Box(q \rightarrow (p \wedge q)) \rightarrow (\Box q \rightarrow \Box(p \wedge q))$
7. $\Box p \rightarrow (\Box q \rightarrow \Box(p \wedge q))$
8. $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$

2. Be careful: $p \not\vdash_K \Box p$. Here is a convincing but incorrect derivation:

1. p (assumption)
2. $\Box p$ (MG)
3. $p \rightarrow \Box p$

3. If $\vdash_{KRZ} \alpha \rightarrow \beta$ then $\vdash_K \Diamond\alpha \rightarrow \Diamond\beta$.

1. $\alpha \rightarrow \beta$
2. $(\alpha \rightarrow \beta) \rightarrow (\sim\beta \rightarrow \sim\alpha)$
3. $\sim\beta \rightarrow \sim\alpha$
4. $\Box(\sim\beta \rightarrow \sim\alpha)$
5. $\Box(\sim\beta \rightarrow \sim\alpha) \rightarrow (\Box\sim\beta \rightarrow \Box\sim\alpha)$
6. $\Box\sim\beta \rightarrow \Box\sim\alpha$
7. $\sim\Box\sim\alpha \rightarrow \sim\Box\sim\beta$
8. $\Diamond\alpha \rightarrow \Diamond\beta$

4. $\vdash_K \Diamond\sim\sim p \rightarrow \Diamond p$. (Hint: use exercise no. 3 and that $\sim\sim p \rightarrow p$ is a propositional tautology.)

5. $\vdash_K (\Diamond p \vee \Diamond q) \rightarrow \Diamond(p \vee q)$.

1. $p \rightarrow p \vee q$
2. $q \rightarrow p \vee q$
3. $\Diamond p \rightarrow \Diamond(p \vee q)$
4. $\Diamond q \rightarrow \Diamond(p \vee q)$
5. $(\Diamond p \vee \Diamond q) \rightarrow \Diamond(p \vee q)$

6. $\vdash_K \Box p \rightarrow \Box(p \rightarrow q)$. (Hint: Use the tautology $p \rightarrow (q \rightarrow p)$, then MG, then K and then MP).