(1) Give the definitions of the following concepts: model for a first-order language; valuation; satisfaction of formulas (atomic and compound) in a model and with respect to a valuation; truth of a formula in a model; when a formula is a semantic consequence of a set of formulas.

Formalize the following sentences in first-order logic and check whether the premises imply the conclusion. If not, give a refuting model.

Premises

Every dog is a mammal.

No mammal is a fish.

Some animals are fish.

Conclusion

Some animals are not dogs.

(2) Give the definitions of: valuation in propositional logic (KRZ); frame and valuation in intuitionistic logic (INT); logical truth in KRZ and INT.

Please check whether the sentence

> "If it is not true that (I ate lunch and drank coffee), then (I did not eat lunch or drink coffee)"

is a logical truth of KRZ and INT. If the answer is no, please provide a refuting interpretation.

(3) The following formulas are given:

a. $(\neg p \to \neg q) \to (p \to q)$

b. $(p \to (\neg q \to \neg r)) \to ((p \wedge \neg q) \to \neg r)$

c. $\neg((p \to \neg p) \wedge (\neg p \to p))$

For each formula that is a theorem of propositional logic, give a derivation. For each formula that is not a theorem of propositional logic, please justify why it is not so.

(4) Give the definitions of: model of $S4$; tautology in $S4$. Check whether the following formulas are tautologies and if not, give an $S4$-model that refutes the formula.

a. $\Diamond p \vee \Diamond \neg p$

b. $\Diamond(p \vee \neg p)$

$\boxed{\text{AS FOR THE DEFINITIONS, THEOREMS: SEE THE LECTURES}}$

① $\forall x (D(x) \to M(x))$ D - dog, M - mammal, F - fish, A - animal

$\forall x (M(x) \to \neg F(x))$ OR: $\neg \exists x (M(x) \land F(x))$

$\exists x (A(x) \land F(x))$

$\exists x (A(x) \land \neg D(x))$

DERIVATION

1. $\exists x (A(x) \land F(x))$ Assumption
2. $A(c) \land F(c)$ c IS A BRAND NEW CONSTANT ; FROM 1.
3. $\forall x (D(x) \to M(x))$ Assumption
4. $\forall x (D(x) \to M(x)) \to (D(c) \to M(c))$ RULE $\forall x \alpha \to \alpha(x/c)$
5. $D(c) \to M(c)$ MP: 3,4
6. $\forall x (M(x) \to \neg F(x))$ Assumption
7. $\forall x (M(x) \to \neg F(x)) \to (M(c) \to \neg F(c))$ RULE: $\forall x \alpha \to \alpha(x/c)$
8. $M(c) \to \neg F(c)$ MP: 6,7
9. $F(c) \to \neg M(c)$ PROP. LOGIC: 8 $[\varphi \to \psi \equiv \neg\psi \to \neg\varphi]$
10. $F(c)$ PROP. LOGIC: 2 $[\varphi \land \psi \vdash \varphi]$
11. $\neg M(c)$ MP: 9, 10
12. $\neg M(c) \to \neg D(c)$ PROP. LOGIC: 8
13. $\neg D(c)$ MP: 11, 12
14. $A(c)$ PROP. LOGIC: 2
15. $A(c) \land \neg D(c)$ PROP. LOGIC: 13, 14
16. $\exists x (A(x) \land \neg D(x))$ RULE: $\varphi(c) \to \exists x \varphi(x)$ 15

SEMANTICALLY: TAKE A MODEL $\mathcal{B} = (B, D^{\mathcal{B}}, M^{\mathcal{B}}, F^{\mathcal{B}}, A^{\mathcal{B}})$, WHERE $D^{\mathcal{B}}, M^{\mathcal{B}}, F^{\mathcal{B}}, A^{\mathcal{B}} \subseteq B$ UNARY RELATIONS

ASSUME $\mathcal{B} \models \forall x (D(x) \to M(x)), \forall x (M(x) \to \neg F(x)), \exists x (A(x) \land F(x))$

THEN $D^{\mathcal{B}} \subseteq M^{\mathcal{B}}$, $M^{\mathcal{B}} \subseteq B \setminus F^{\mathcal{B}}$ AND $\boxed{\text{THERE EXISTS } c \in A^{\mathcal{B}} \cap F^{\mathcal{B}}}$

WE NEED: THERE EXIST AN ELEMENT OF $A^{\mathcal{B}} \cap (B \setminus D^{\mathcal{B}})$ (i.e. THIS SET IS NOT EMPTY)

BUT THE ELEMENT c IS SUCH:

$c \in A^{\mathcal{B}} \cap F^{\mathcal{B}}$ AND $D^{\mathcal{B}} \subseteq M^{\mathcal{B}} \subseteq (B \setminus F^{\mathcal{B}})$, THUS $D^{\mathcal{B}} \subseteq B \setminus F^{\mathcal{B}}$ THUS $F^{\mathcal{B}} \subseteq B \setminus D^{\mathcal{B}}$

SO, IF $c \in F^{\mathcal{B}}$, THEN $c \in B \setminus D^{\mathcal{B}}$

$\Rightarrow \boxed{c \in A^{\mathcal{B}} \cap (B \setminus D^{\mathcal{B}})}$

② FORMALIZATION: $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$

METHODS IN KRZ:

ⓘ TRUTH TABLE

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$
0	0	0	1	1	1	1	1
1	0	0	1	0	1	1	1
0	1	0	1	1	0	1	1
1	1	1	0	0	0	0	1

TAUTOLOGY

(ii) TREES:

$\neg(\neg(p \wedge q) \rightarrow (\neg p \vee \neg q))$

$\neg(p \wedge q)$

$\neg(\neg p \vee \neg q)$

$\neg\neg p$

$\neg\neg q$

p

q

$\neg p$ ⊗ | $\neg q$ ⊗

ALL BRANCHES ARE CLOSED $\Rightarrow$ TAUTOLOGY

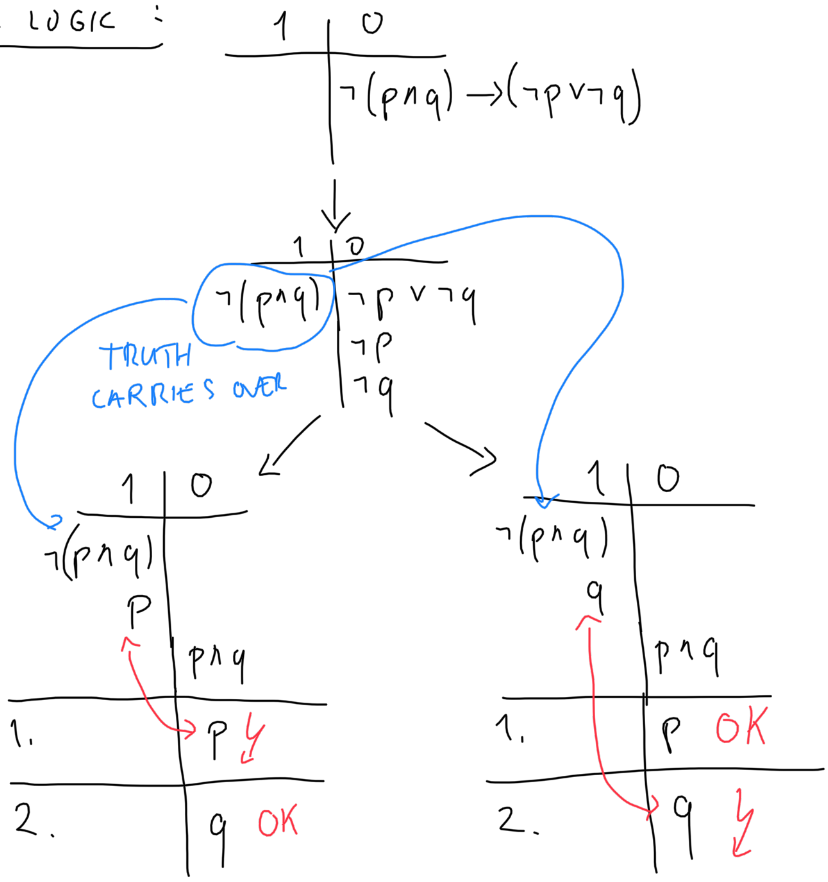
(iii) TABLES:

1	0
	$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$
$\neg(p \wedge q)$	$\neg p \vee \neg q$
	$p \wedge q$
	$\neg p$
	$\neg q$
p	
q	
1.	p
2.	q

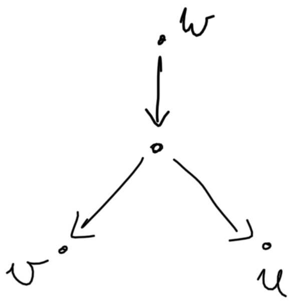
ALL CASES LED TO CONTRADICTION

$\Rightarrow$ TAUTOLOGY

IN INTUITIONISTIC LOGIC:



COUNTERMODEL:



$v(q) = 0$

$u(p) = 0$

$w(\neg(p \wedge q)) = 1$

$w(\neg p \vee \neg q) = 0$

③ (i) $(\neg p \to \neg q) \to (p \to q)$

NOT A THEOREM, SHOWN BY THE EVALUATION $p = 1$, $q = 0$

$$(\neg 1 \to \neg 0) \to (1 \to 0)$$
$$(0 \to 1) \to (1 \to 0)$$
$$1 \to 0$$
$$0$$

(ii) THE FORMULA HAS THE FORM $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$ WITH $\alpha = p$, $\beta = \neg q$, $\gamma = \neg r$

NATURAL DEDUCTION:

1.	$\alpha \to (\beta \to \gamma)$	CONDITIONAL PROOF STARTS
2.	$\alpha \wedge \beta$	COND. PROOF STARTS
3.	α	FROM 2, $\frac{\alpha \wedge \beta}{\alpha}$
4.	β	FROM 2 $\frac{\alpha \wedge \beta}{\beta}$
5.	$\beta \to \gamma$	MP: 1,3
6.	γ	MP: 4,5
7.	$\alpha \wedge \beta \to \gamma$	COND. PROOF ENDS
8.	$(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$	COND. PROOF ENDS

HILBERT STYLE:

$$\vdash (\alpha \to (\beta \to \gamma)) \to (\alpha \wedge \beta \to \gamma)$$

$\Updownarrow$ DEDUCTION THM

$$\alpha \to (\beta \to \gamma) \vdash (\alpha \wedge \beta) \to \gamma$$

$\Updownarrow$ DEDUCTION THM

$$\alpha \to (\beta \to \gamma), \alpha \wedge \beta \vdash \gamma$$ ← LET'S SHOW THIS ONE

1.	$\alpha \to (\beta \to \gamma)$	ASSUMPTION
2.	$\alpha \wedge \beta$	ASSUMPTION
3.	$\alpha \wedge \beta \to \alpha$	AXIOM
4.	$\alpha \wedge \beta \to \beta$	AXIOM
5.	α	MP: 2,3
6.	β	MP: 2,4
7.	$\beta \to \gamma$	MP: 1,5
8.	$\boxed{\gamma}$	MP: 6,7

(iii) NATURAL DEDUCTION:

1.	$(p \to \neg p) \wedge (\neg p \to p)$	CONDITIONAL PROOF
2.	$p \to \neg p$	$\wedge$-ELIM: 1
3.	$\neg p \to p$	$\wedge$-ELIM: 1
4.	p	CONDITIONAL PROOF
5.	$\neg p$	MP: 4, 2
6.	$\perp$	CONTRADICTION: 4, 5
7.	$\neg p$	REDUCTIO AD ABS: 4–6
8.	p	MP: 7, 3
9.	$\perp$	CONTRADICTION: 7, 8
10.	$\neg\big((p \to \neg p) \wedge (\neg p \to p)\big)$	REDUCTIO AD ABS. 1–9

HILBERT STYLE:

LEMMA: $\vdash (p \to \neg p) \to \neg p$

PROOF: $\overset{\text{DEDUCTION THM}}{\iff}$ $p \to \neg p \vdash \neg p$

$\iff \{p \to \neg p, p\}$ CONTRADICTORY

1.	p	ASS.
2.	$p \to \neg p$	ASS.
3.	$\neg p$	MP: 1, 2

CONTRADICTION: 1, 3

LEMMA: $\vdash (\neg p \to p) \to p$ PROOF: SIMILAR

BACK TO THE EXERCISE:

$$\vdash \neg\big((p \to \neg p) \wedge (\neg p \to p)\big)$$

$$\Updownarrow$$

$(p \to \neg p) \wedge (\neg p \to p)$ IS CONTRADICTORY $(\ldots \vdash \perp)$

1.	$(p \to \neg p) \wedge (\neg p \to p)$	ASSUMPTION
2.	$(p \to \neg p) \wedge (\neg p \to p) \to (p \to \neg p)$	AXIOM $\alpha \wedge \beta \to \alpha$
3.	$(p \to \neg p) \wedge (\neg p \to p) \to (\neg p \to p)$	AXIOM $\alpha \wedge \beta \to \beta$
4.	$p \to \neg p$	MP: 1, 2
5.	$\neg p \to p$	MP: 1, 3
6.	$(p \to \neg p) \to \neg p$	LEMMA } (SUBPROOFS...)
7.	$(\neg p \to p) \to p$	LEMMA
8.	p	MP: 4, 6
9.	$\neg p$	MP: 5, 7
10.	$p \wedge \neg p$	CONTRADICTION

4. (i)

1	0
	$\Diamond p \vee \Diamond \neg p$
	$\Diamond p$
	$\Diamond \neg p$
	p
	$\neg p$
p	

$\lightning$

TAUTOLOGY OF S4

NOTE: WE USED REFLEXIVITY

(ii)

1	0
	$\Diamond (p \vee \neg p)$
	$p \vee \neg p$
	p
	$\neg p$
p	

$\lightning$

TAUTOLOGY OF S4

NOTE: WE USED REFLEXIVITY