Axioms:

Ax.1. $\alpha \to (\beta \to \beta)$

Ax.2. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

Ax.3. $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$

Ax.4. $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$

Ax.5. $(\alpha \wedge \beta) \to \alpha$

Ax.6. $(\alpha \wedge \beta) \to \beta$

Ax.7. $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$

Ax.8. $\alpha \to (\alpha \vee \beta)$

Ax.9. $\beta \to (\alpha \vee \beta)$

Ax.10. $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

Ax.11. $(\sim\alpha \to \sim\beta) \to (\beta \to \alpha)$

Inference rule: $\alpha,\ \alpha \to \beta \ \vdash \ \beta$ (RO) (Modus Ponens)

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1. Do the following substitutions.
   - $e(p) = p \to p, \quad e(q) = p \wedge q, \quad \phi = (p \to p) \to q, \quad h^e(\phi) = ?$
   - $e(p) = q, \quad e(q) = p, \quad \phi = r \to (p \wedge q), \quad h^e(\phi) = ?$
   - $e(p) = r \to q, \quad e(q) = r \wedge p \to q, \quad \phi = (p \wedge (p \to q)) \to q, \quad h^e(\phi) = ?$

   Check whether the formula $\big((r \to q) \wedge ((r \to q) \to (r \wedge p \to q))\big) \to (r \wedge p \to q)$ is a tautology.

2. $\vdash \alpha \to \alpha$
   1. $(\alpha \to (\beta \to \beta)) \to (\alpha \to \alpha)$ Ax. 1
   2. $\alpha \to (\beta \to \beta)$ Ax. 1
   3. $\alpha \to \alpha$ RO

3. $\vdash \alpha \to (\beta \to \alpha)$
   1. $\beta \to (\alpha \to \alpha)$ Ax. 1
   2. $(\beta \to (\alpha \to \alpha)) \to (\alpha \to (\beta \to \alpha))$ Ax. 3
   3. $\alpha \to (\beta \to \alpha)$ RO

4. $\alpha \to \beta,\ \beta \to \gamma \quad \vdash \quad \alpha \to \gamma$ RPI
   1. $\alpha \to \beta$
   2. $\beta \to \gamma$
   3. $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ Ax. 2
   4. $(\beta \to \gamma) \to (\alpha \to \gamma)$ RO
   5. $\alpha \to \gamma$ RO

5. Show that $\alpha \to (\beta \to \gamma) \vdash \beta \to (\alpha \to \gamma)$. (RKom)

6. Show that $\alpha \vdash \beta \to \alpha$ (Hint: use Ax.1 and Ax.2)

Deduction: $\Phi, \alpha \ \vdash \ \beta$ if and only if $\Phi \ \vdash \ \alpha \to \beta$
Theorem: $\Phi \ \vdash \ \alpha$ if and only if $\Phi \cup \{\sim\alpha\}$ is contradictory.
Similarly: $\Phi \ \vdash \sim\alpha$ if and only if $\Phi \cup \{\alpha\}$ is contradictory.

7. Assume $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$. Is it true in this case that $\vdash \varphi$ or $\vdash \psi$?

8. Show $\alpha \to \beta,\ \beta \to \gamma \ \vdash \ \alpha \to \gamma$ using the deduction theorem.

9. Show that $\alpha \to \beta, \gamma \ \nvdash \ \beta$.

10. Prove that if $\Sigma \vdash \varphi$ then there is a finite subset $\Gamma \subseteq \Sigma$ such that $\Gamma \vdash \varphi$.

11. Show $\alpha \to \beta, \sim\beta \ \vdash \sim\alpha$

12. Show that if $\alpha \vdash \varphi$ and $\sim\alpha \vdash \varphi$, then φ is a theorem (i.e. $\vdash \varphi$).

13. Is it true that if $\neg\alpha \vdash \beta$ then $\neg\beta \vdash \alpha$?

14. Assume $\alpha \vdash \beta$ and $\vdash \alpha$. Show that $\vdash \beta$.