For a formula $\varphi$, variable $x$ and term $t$, $\varphi[x/t]$ is the formula obtained from $\varphi$ by replacing each free occurrence of $x$ with $t$.

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**1.** Consider the following formulas.

(a) $\alpha_1 = P(x) \to \exists y P(y)$

(b) $\alpha_2 = \forall x P(x) \to P(y)$

(c) $\alpha_3 = \exists x(P(x) \to Q(x, y))$

(c) $\alpha_4 = \exists x(P(x) \vee Q(x)) \to (\exists x P(x) \vee \exists x Q(x))$

(c) $\alpha_5 = \forall x(P(x) \wedge Q(y)) \to (\exists y R(x, y, z) \wedge \forall z R(z, y, y))$

In each case do the substitutions $\alpha_i[x/y]$, $\alpha_i[y/x]$, $\alpha_i[x/t]$, where $t = P(x)$.

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*Admissible substitutions* (dozwolone podstawienia) are defined by recursion on the complexity of $\varphi$ as follows.

- if $\varphi$ is atomic, then $\varphi[x/t]$ is admissible,
- $(\varphi \vee \psi)[x/t]$ is admissible if both $\varphi[x/t]$ and $\psi[x/t]$ are admissible,
- $(\sim\varphi)[x/t]$ is admissible if $\varphi[x/t]$ is such,
- $(\exists x\varphi)[x/t]$ is admissible (note that the substitution gives $\exists x\varphi$),
- $(\exists y\varphi)[x/t]$ is admissible if $x$ and $y$ are different, $\varphi[x/t]$ is admissible, and either $x$ is not free in $\varphi$ or $y$ does not occur in $t$.

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**2.** Select the admissible substitutions.

(a) $\forall P(y) \to P(x)\ [x/f(x, y)]$

(b) $\exists x \forall y(R(f(y, x), z)) \wedge P(y))\ [y/z]$

(c) $\exists x(R(c, f(x, y)))\ [y/g(x, y)]$

(d) $\forall x P(x) \vee \exists z R(f(z, y), x)\ [y/g(f(x, y), y)]$

(e) $\forall x \exists y(R((fx, y), z) \wedge P(x))\ [z/c]$

**3.** Give example for a formula $\varphi$ and term $t$ such that $\models \varphi$ while $\not\models \varphi[x/t]$.

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**Substitution Theorem:** If $\varphi[x/t]$ is an admissible, then

$$\mathcal{A} \models (\varphi[x/t])[e] \quad \text{if and only if} \quad \mathcal{A} \models \varphi[e, x = t^{\mathcal{A}}[e])].$$

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**4.** Let $\varphi$ be a formula, $t$ be a term and $e$ be an evaluation over $\mathcal{A}$. Give examples for all possible truth-values of $\mathcal{A} \models \varphi(x/t)[e]$ and $\mathcal{A} \models \varphi[e, x = t^{\mathcal{A}}[e])]$ when the substitution is

(a) admissible

(b) not admissible.