- Two sets have the same "size" (=cardinality) if there is a bijection between them (=we can pair the elements). Notation: $|A| = |B|$.
- **Finite:** A is finite, if $|A| = n$ for some natural number n
- **Countably infinite:** A is countably infinite, if $|A| = |\mathbb{N}|$
- **Countable:** A is countable, if it is finite or countably infinite.

Important theorems that we discuss:

- $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$,
- **CCC-theorem:** A countable union of countable sets is countable.
- $|\mathbb{N}| \neq |\mathbb{R}|$ ($\rightarrow$ there are different sizes of infinite)
- $|A| \neq |\mathcal{P}(A)|$ ($\rightarrow$ there are "infinitely many" different infinities)

Exercises:

1. **Hilbert-hotel:** Up in the valleys of the Hilbertowo mountains there is a hotel with countably infinitely many rooms. The rooms are numbered by the natural numbers. On a busy weekend (people visit the surroundings to hazard money in Settingham, see below) all the rooms are occupied. But Saturday morning a bus of new guests arrive. What can the receptionist do to accommodate them? (The buses can carry countably many passengers).

2. In a dusty pub of Settingham, there is an old slot machine that went wrong: if you feed it with one złoty, it gives countably many złotych back. We start with a złoty and in each turn we can feed the machine with one of the returned ones. Prove that we can play so that after countably many steps we will have no money at all. How should we play if we would like to have exactly 17 złotych at the end?

3. Prove that if A is countable then so is A^n, where n is an arbitrary, fixed natural number.

4. Let A be a countable set and suppose B contains all of the finite sequences whose elements are in A. Prove that B is countable, as well.

5. **The flea-game:**

   (a) Florek the flea choses his secret natural number and starting from the origin, in each turn he adds this number to his current position and jumps there. We would like to catch the flea. In each turn we can guess his position. Show that there is a strategy with which we can always catch him in a finite number of steps.

   (b) What to do if Florek is jumping not on the line, but on the 2D plane?

   (c) The same as in **(a)** but instead of choosing a secret vector, the Florek choses an algorithm. In the n^{th} round he feeds the algorithm with n as an input and jumps to the output's coordinates.

6. Prove that the following sets have the same cardinality. Construct a bijection in each case.

   (a) $\mathbb{Q}$ and $[a, b] \cap \mathbb{Q}$. **(b)** $[0, 1]$ and $[a, b]$. **(c)** $(0, 1)$ and $[0, 1]$. **(d)** $(0, \infty)$ and $(-\infty, \infty)$.

   (e) $(-\infty, \infty)$ and $(-\pi, \pi)$. **(f)** $\mathbb{R}$ and $(0, 1)$. **(g)** $(0, 1]$ and $(0, 1)$. **(h)** $[0, 1]$ and $[0, 1)$.

   (i) $\mathcal{P}(\omega)$ and $[0, 1]$. **(j)** $\mathbb{R}$ and $[0, 1)$. **(k)** $\mathcal{P}(\omega)$ and $\mathbb{R}$.