**Derivations.** Briefly: use anything from propositional logic + new rules (that we discuss one by one). The first such new rule is:

> $\forall Out$: If $x$ is **free** in $\varphi(x)$, then
>
> $$\frac{\forall x \varphi(x)}{\varphi[x/t]}$$

(1) Let $c$ be a constant. Check if the $\forall Out$ rule can be applied. If so, what is the result?

(a) $\forall x F(x)$

(b) $\forall x(F(x) \to G(x))$

(c) $\forall x(F(x) \to \forall x G(x))$

(d) $\sim\forall x G(x)$

(e) $\forall x F(x) \to \forall x G(x)$

(f) $\forall x \exists y R(x, y)$

(g) $\forall x R(x, x)$

(2) Construct derivations using the $\forall Out$ rule.

$$\begin{array}{l} \forall x(F(x) \to H(x)) \\ F(c) \\ \hline H(c) \end{array} \qquad \begin{array}{l} \forall x(S(x) \to P(x)) \\ \forall x((S(x) \wedge P(x)) \to D(x)) \\ S(m) \\ \hline D(m) \end{array}$$

(3) Derive

$$\begin{array}{l} \forall x F(x) \to \forall x G(x) \\ \hline \forall x F(x) \to G(a) \end{array} \qquad \begin{array}{l} \forall x(F(x) \to H(x)) \\ \sim H(b) \\ \hline \sim\forall x F(x) \end{array} \qquad \begin{array}{l} \forall x(F(x) \to \forall y R(x, y)) \\ \forall x \forall y(R(x, y) \to \forall z G(z)) \\ \sim G(b) \\ \hline \sim F(a) \end{array}$$

> $\exists In$: If $\varphi[x/t]$ is a substitution instance of $\varphi(x)$, then
>
> $$\frac{\varphi[x/t]}{\exists x \varphi(x)}$$

(4) Apply the $\exists In$ rule in all possible ways to the formulas below (if possible).

(a) $F(b)$

(b) $R(c, d)$

(c) $F(c) \wedge G(c)$

(5) Derive

$$\begin{array}{l} \forall x(F(x) \to H(x)) \\ F(a) \\ \hline \exists x H(x) \end{array} \qquad \begin{array}{l} \forall x(G(x) \to H(x)) \\ G(b) \\ \hline \exists x(G(x) \wedge H(x)) \end{array} \qquad \begin{array}{l} \exists x \sim R(x, a) \to \sim\exists x R(a, x) \\ \sim R(a, a) \\ \hline \sim R(a, b) \end{array}$$

$K$:

$$\frac{\forall x(\varphi(x) \to \psi(x))}{\forall x\varphi(x) \to \forall x\psi(x)}$$

*Gen*: If $x$ is *not* free in $\varphi$, then

$$\frac{\varphi}{\forall x\varphi(x)}$$

(6) Derive

$$\frac{\forall x(F(x) \to G(x)) \quad \forall xF(x)}{\forall xG(x)} \qquad \frac{\forall x(F(x) \to G(x)) \quad \forall x(G(x) \to H(x))}{\forall x(F(x) \to H(x))}$$

(7) Which of the following formulas are logical truth?

(a) $\exists x\forall y\varphi \to \forall y\exists x\varphi$

(b) $\forall x\exists y\varphi \to \exists y\forall x\varphi$

(c) $\forall x\exists x\varphi \to \exists x\forall x\varphi$

(8) Formalize the sentences below (in first-order logic). Check whether the premises imply the conclusions, and if so, construct a derivation.

**a. Premises**: Every freshman is happy. Jake is a freshman.

**a. Conclusion**: Jake is happy.

---

**b. Premises**: Every snake is poisonous. Every poisonous snake is dangerous. Joe is a snake.

**b. Conclusion**: Joe is dangerous.

---

**c. Premises**: Every dog is a mammal. No mammal is a fish. Some animals are fish.

**c. Conclusion**: Some animals are not dogs.

## Aksjomaty

Niech α i β będą formułami; $x, y, z, x_1,\ldots, x_n, y_1,\ldots, y_n$ zmiennymi indywiduowymi; $R$ $n$-argumentowym symbolem relacyjnym; $f$ $n$-argumentowym symbolem funkcyjnym; zaś $t$ termem.

| | | | |
|---|---|---|---|
| 1. | Podstawienia aksjomatów KRZ | | |
| 2. | $\forall x\,(\alpha \to \beta) \to (\forall x\,\alpha \to \forall x\,\beta)$ | | **rozdzielność ∀ względem →** |
| 3. | $\forall x\,\alpha \to \alpha(x/t)$, | jeśli tylko Podst(α, $t$, $x$) | **podstawienie** |
| 4. | $\alpha \to \forall x\,\alpha$, | jeśli tylko $x$ nie jest wolna w α | **generalizacja** |
| 5. | $x = x$<br>$(x = y) \to (y = x)$<br>$((x = y) \wedge (y = z)) \to (x = z)$ | | **aksjomaty równości** |
| 6. | $((x_1 = y_1) \wedge\ldots\wedge (x_n = y_n)) \to (R(x_1,\ldots, x_n) \leftrightarrow R(y_1,\ldots, y_n))$ | | **zgodność relacji z równością** |
| 7. | $((x_1 = y_1) \wedge\ldots\wedge (x_n = y_n)) \to (f(x_1,\ldots, x_n) = f(y_1,\ldots, y_n))$ | | **zgodność funkcji z równością** |