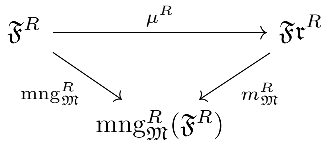
Clarifications concerning Definition 3.3.27(6):

For any set $R \in Sig$ let $\mathfrak{Fr}^R$ denote the $\mathrm{Alg}_m(\mathcal{L}^R)$-free algebra generated by R. Let $\mu^R : \mathfrak{F}^R \to \mathfrak{Fr}^R$ be the homomorphic extension of the identity mapping $id : R \to R$. As $\mathfrak{Fr}^R$ has the universal mapping property with respect to the class $\mathrm{Alg}_m(\mathcal{L}^R)$, for any model $\mathfrak{M} \in M^R$ there is a homomorphism $m^R_{\mathfrak{M}} : \mathfrak{Fr}^R \to \mathrm{mng}^R_{\mathfrak{M}}(\mathfrak{F}^R)$ such that $m^R_{\mathfrak{M}} \circ \mu^R = \mathrm{mng}^R_{\mathfrak{M}}$, i.e. the diagram below commutes.



Define the tautological congruence $\backsim^R$ *of* $\mathfrak{Fr}^R$ by writing[1]

$$\mu^R(\varphi) \backsim^R \mu^R(\psi) \ \text{ if and only if } \ (\forall \mathfrak{M} \in M^R)\ m^R_{\mathfrak{M}}(\mu^R(\varphi)) = m^R_{\mathfrak{M}}(\mu^R(\psi)), \qquad (1)$$

that is, $\backsim^R$ is $\bigcap_{\mathfrak{M} \in M^R} \ker(m^R_{\mathfrak{M}})$. Notice that $\varphi \sim^R \psi$ if and only if $\mu^R(\varphi) \backsim^R \mu^R(\psi)$. [2]

Let now $P_i \in Sig$ be disjoint sets for $i \in I$ and let P be the disjoint union $\bigcup_{i \in I} P_i$. The congruences $\backsim^{P_i}$ are relations in $\mathfrak{Fr}^P$. Item (6) of Definition 3.3.27 requires that

$$\backsim^P = \mathrm{Cg}^{\mathfrak{Fr}^P}\Big(\bigcup_{i \in I} \backsim^{P_i}\Big) \qquad (2)$$

holds.

[1] Observe the difference between the symbols $\sim$ and $\backsim$.

[2] An alternative way would be to define $\backsim^R$ as the μ^R-image of $\sim^R$, that is, $\backsim^R = \{\langle \mu^R(\varphi), \mu^R(\psi)\rangle : \varphi \sim^R \psi\}$. As $\mathfrak{Fr}^R$ is the $\mathrm{Alg}_m(\mathcal{L}^R)$-free algebra, and $\sim^R$ was defined by the intersection of kernels of meaning homomorphisms, this yields the same definition. Note, however, that the surjective homomorphic image of a congruence is not necessarily a congruence, as transitivity can be violated. In general, surjective homomorphic images of congruences are only tolerance relations: reflexive, symmetric and compatible.