0.1 Lemma. For $Q \subseteq P$ in Sig we have $\mathrm{Alg}_{\models}(\mathcal{L}^Q) \subseteq \mathbf{HS}\mathrm{Alg}_{\models}(\mathcal{L}^P)$

Proof. For any $\mathfrak{A} \in \mathrm{Alg}_{\models}(\mathcal{L}^Q)$ there is a set $K \subseteq M^Q$ such that $\mathfrak{A} \cong \mathfrak{F}^Q/\sim_K$. By properties of a sublogic, for each $\mathfrak{M} \in K$ there is $\mathfrak{N} = \mathfrak{N}_{\mathfrak{M}} \in M^P$ such that $\mathrm{mng}^Q_{\mathfrak{M}} = \mathrm{mng}^P_{\mathfrak{N}} \upharpoonright F^Q$. Let

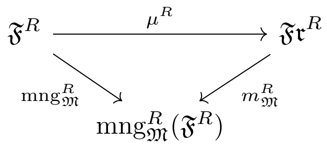
$$L = \{\mathfrak{N}_{\mathfrak{M}} \in M^P :\ \mathfrak{M} \in K\}.$$

For φ, $\psi \in F^Q$ whenever $\varphi \sim_L \psi$ holds then $\varphi \sim_K \psi$ also holds. The set $\{\varphi/\sim_L\colon\ \varphi \in F^Q\}$ is the universe of a subalgebra $\mathfrak{B}$ of $\mathfrak{F}^P/\sim_L$, and the mapping

$$f : \mathfrak{B} \to \mathfrak{F}^Q/\sim_K, \quad f(\varphi/\sim_L) = \varphi/\sim_K$$

is a surjective homomorphism. ∎

For any set $R \in Sig$ let $\mathfrak{Fr}^R$ denote the $\mathrm{Alg}_m(\mathcal{L}^R)$-free algebra generated by R. Let $\mu^R : \mathfrak{F}^R \to \mathfrak{Fr}^R$ be the homomorphic extension of the identity mapping $id : R \to R$. As $\mathfrak{Fr}^R$ has the universal mapping property with respect to the class $\mathrm{Alg}_m(\mathcal{L}^R)$, for any model $\mathfrak{M} \in M^R$ there is a homomorphism $m^R_{\mathfrak{M}} : \mathfrak{Fr}^R \to \mathrm{mng}^R_{\mathfrak{M}}(\mathfrak{F}^R)$ such that $m^R_{\mathfrak{M}} \circ \mu^R = \mathrm{mng}^R_{\mathfrak{M}}$, i.e. the diagram below commutes.



Define the tautological congruence $\backsim^R$ *of* $\mathfrak{Fr}^R$ by writing[1]

$$\mu^R(\varphi) \backsim^R \mu^R(\psi) \ \text{ if and only if } \ (\forall \mathfrak{M} \in M^R)\ m^R_{\mathfrak{M}}(\mu^R(\varphi)) = m^R_{\mathfrak{M}}(\mu^R(\psi)), \qquad (1)$$

that is, $\backsim^R$ is $\bigcap_{\mathfrak{M} \in M^R} \ker(m^R_{\mathfrak{M}})$. Notice that $\varphi \sim^R \psi$ if and only if $\mu^R(\varphi) \backsim^R \mu^R(\psi)$. [2]

0.2 Theorem. Let $\mathfrak{A} \in \mathbf{SP}\mathrm{Alg}_m(\mathbf{L})$, $P_i \in Sig$ disjoint sets and $h_i : \mathfrak{F}^{P_i} \to \mathfrak{A}$ homomorphisms such that $\sim^{P_i} \subseteq \ker(h_i)$. Let P be the disjoint union of the P_i's and h be the (unique) homomorphic extension $h : \mathfrak{F}^P \to \mathfrak{A}$ of the union $\bigcup_i h_i$. Then $\sim^P \subseteq \ker(h)$.

Proof. The range of h_i is a subalgebra of $\mathfrak{A}$ and thus it also belongs to $\mathbf{SP}\mathrm{Alg}_m(\mathbf{L})$. By Claim 3.3.41, there is a class $K_i \subseteq M^{P_i}$ such that $\mathrm{ran}(h_i) \cong \mathfrak{F}^{P_i}/\sim_{K_i}$. As $\mathcal{L}^{P_i}$ is a sublogic of $\mathcal{L}^P$, for each $\mathfrak{M} \in K_i$ there is a model $\mathfrak{N} \in M^P$ such that

$$\mathrm{mng}^{P_i}_{\mathfrak{M}} = \mathrm{mng}^P_{\mathfrak{N}} \upharpoonright F^{P_i}.$$

Let K be the collection of such $\mathfrak{N}$'s for every $\mathfrak{M} \in K_i$ for all i. Then $\mathrm{ran}(h_i) \in \mathbf{HS}(\mathfrak{F}^P/\sim_K)$ (cf. the proof of Lemma 0.1) and as h extends each h_i we also get that

$$\mathrm{ran}(h) \in \mathbf{H}(\mathfrak{F}^P/\sim_K)$$

and therefore $\mathrm{ran}(h) \in \mathbf{HSP}\mathrm{Alg}_m(\mathcal{L}^P)$. The $\mathrm{Alg}_m(\mathcal{L}^P)$-free algebra $\mathfrak{Fr}^P$ is free with respect to the class $\mathbf{HSP}\mathrm{Alg}_m(\mathcal{L}^P)$ as well, therefore, writing $\mathfrak{B} = \mathrm{ran}(h)$, there is a mapping g such that the the diagram below commutes

[1] Observe the difference between the symbols $\sim$ and $\backsim$.

[2] An alternative way would be to define $\backsim^R$ as the μ^R-image of $\sim^R$, that is, $\backsim^R = \{\langle \mu^R(\varphi), \mu^R(\psi)\rangle :\ \varphi \sim^R \psi\}$. As $\mathfrak{Fr}^R$ is the $\mathrm{Alg}_m(\mathcal{L}^R)$-free algebra, and $\sim^R$ was defined by the intersection of kernels of meaning homomorphisms, this yields the same definition. Note, however, that the surjective homomorphic image of a congruence is not necessarily a congruence, as transitivity can be violated. In general, surjective homomorphic images of congruences are only tolerance relations: reflexive, symmetric and compatible.

$$\begin{array}{ccc} \mathfrak{F}^P & \xrightarrow{\mu^P} & \mathfrak{Fr}^P \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & \mathfrak{B} & \end{array}$$

By 3.3.27(6) the tautological congruence $\backsim^P$ of $\mathfrak{Fr}^P$ is generated by the union of the congruences $\backsim^{P_i}$ (these are congruences of $\mathfrak{Fr}^{P_i}$ and hence relations in $\mathfrak{Fr}^P$):

$$\backsim^P = \mathrm{Cg}^{\mathfrak{Fr}^P}\Big(\bigcup_{i\in I} \backsim^{P_i}\Big) \tag{2}$$

By commutativity of the diagram we have that

$$\sim^P \subseteq \ker(h) \quad \text{iff} \quad \backsim^P \subseteq \ker(g).$$

But this latter inclusion follows as we assumed that for each i $\sim^{P_i} \subseteq \ker(h_i)$ (and thus $\backsim^{P_i} \subseteq \ker(g)$) holds.

∎