Two questions of Kowalski–Słomczyńska

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Abstract. That the inclusion $Q(F_3(X)) \subseteq \mathsf{Pa}_3^-$ is proper and $\mathrm{Mod}(\mathbf{qb}_3)$ is not generated by free p-algebras is proved, answering two open questions from Kowalski–Słomczyńska [1].

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Following the framework and notation of Kowalski and Słomczyńska [1], we omit formal definitions here for the sake of brevity. All terminology used in this note is introduced in their original work.

Let P be the poset of height 2 with four maximal elements $\{a, b, c, d\}$ and minimal elements t_{abc}, t_{abd}, t_{acd}, and t_{bcd} with order relations $t_{uvw} \leq u, v, w$ and no other elements compared. Let $\mathbf{A} = \varepsilon(P)$ be the dual algebra. For an element x of a poset, $\mathrm{M}(x)$ denotes the set of maximal elements above x.

Lemma 1. $\mathbf{A} \in \mathsf{Pa}_3$.

Proof. As $\mathsf{Pa}_3 = V(\bar{\mathbf{B}}_3)$ it is enough to show that A embeds into $\bar{\mathbf{B}}_3^4$. By duality, this amounts to constructing a pp-morphism $\biguplus_{i=1}^4 \delta(\bar{\mathbf{B}}_3) \twoheadrightarrow P$. For each minimal element t of P let S_t be a copy of $\delta(\bar{\mathbf{B}}_3)$. S_t has 3 maximal elements and one bottom element below all three. Define a map $g : \uplus_t S_t \twoheadrightarrow P$ as follows: on the copy $S_{t_{abc}}$ map its three maximal elements bijectively to a, b, c, and its bottom element to t_{abc}; similarly for the other S_t's. Then g is a surjective pp-morphism: it is clearly order-preserving, and for each bottom element we have that the image of its three maximal extensions is exactly the set of maximal extensions of the corresponding t_{abc}. □

Lemma 2. $\mathbf{A} \notin Q(\mathcal{F})$*, where* $\mathcal{F}$ *is the class of free p-algebras. In particular,* $\mathbf{A} \notin Q(\mathbf{F}_3(X))$ *for any* X*.*

Proof. For a contradiction, suppose $\mathbf{A} \in Q(\mathcal{F})$. Because $\mathbf{A}$ is finite, [1, Lemma 1.6] gives $\mathbf{A} \in \mathbf{ISP}_{fin}(\mathcal{F})$, and so $\mathbf{A} \leq \Pi_i \mathcal{F}_i(X_i)$, where $\mathcal{F}_i(X_i)$ is a free p-algebra generated by the set X_i. Let $\pi_i : \mathbf{A} \to \mathcal{F}_i(X_i)$ be the

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coordinate projections. Since $\mathbf{A}$ is finite, each image $\pi_i(\mathbf{A})$ is a finite subalgebra of $\mathcal{F}_i(X_i)$, hence it is contained in some finitely generated $\mathcal{F}_i(k_i)$. By [1, Lemma 2.5] finitely generated p-algebras are of the form $\mathbf{F}_{m_i}(k_i)$, thus $\mathbf{A} \leq \prod_i \mathbf{F}_{m_i}(k_i)$. By duality [1, Lemma 1.7]) there is a surjective pp-morphism

$$g : \biguplus_i P_i \twoheadrightarrow P, \qquad \text{where } P_i = \delta(F_{m_i}(k_i)).$$

Pick the point $t_{abc} \in P$ and choose i and $y \in P_i$ such that $g(y) = t_{abc}$. The pp-condition of g gives $\mathrm{M}(g(y)) = g(\mathrm{M}(y))$. But $\mathrm{M}(g(y)) = \mathrm{M}(t_{abc}) = \{a, b, c\}$, so there exist maximal elements $\alpha, \beta, \gamma \in \mathrm{M}(y)$ such that $g(\alpha) = a$, $g(\beta) = b$, $g(\gamma) = c$. It follows from [1, Lemma 2.5(1)]) that $k_i \geq 2$. Also, $m_i \geq 2$, because in $\mathbf{F}_1(k_i)$ there is no element y with $|\,\mathrm{M}(y)| = 3$. Now apply [1, Lemma 2.5(2)] inside P_i: every nonempty set of maxima of size $\leq m_i$ occurs as $\mathrm{M}(z)$ for some $z \in P_i$. In particular, there exists $z \in P_i$ such that $\mathrm{M}(z) = \{\alpha, \beta\}$. By the pp-condition it follows that

$$\mathrm{M}(g(z)) = g(\mathrm{M}(z)) = \{g(\alpha), g(\beta)\} = \{a, b\}.$$

But this is impossible in P: there is no $x \in P$ with $\mathrm{M}(x) = \{a, b\}$. □

Lemma 3. $\mathbf{A} \in \mathrm{Mod}(\mathbf{qb}_3)$.

Proof. By duality, $\mathbf{A} \models \mathbf{qb}_3$ iff there is no surjective pp-morphism $P \twoheadrightarrow \delta(\bar{\mathbf{B}}_3)$ (see [1, Lemma 2.3]). By way of contradiction, suppose $f : P \twoheadrightarrow \delta(\bar{\mathbf{B}}_3)$ is a surjective pp-morphism. Since P has four maximal elements and $\delta(\bar{\mathbf{B}}_3)$ has three, two distinct maximum elements, say a, b, must satisfy $f(a) = f(b)$. Since f is surjective on maxima, pick a third maximal element, say c, with $f(c) \neq f(a)$. Consider t_{abc}. Then $\mathrm{M}(t_{abc}) = \{a, b, c\}$ and by the pp-condition

$$\mathrm{M}(f(t_{abc})) = f(\mathrm{M}(t_{abc})) = \{f(a), f(b), f(c)\} = \{f(a), f(c)\}$$

is a 2-element set. But in $\delta(\bar{\mathbf{B}}_3)$ there is no element x with $|\,\mathrm{M}(x)| = 2$. Hence no such f exists and $A \models \mathbf{qb}_3$. □

Recall $\mathsf{Pa}_3^- = \mathsf{Pa}_3 \cap \mathrm{Mod}(qb_3)$.

Theorem 4. *The inclusion $Q(F_3(X)) \subsetneq \mathsf{Pa}_3^-$ is proper, and $\mathrm{Mod}(\mathbf{qb}_3)$ is not generated by free p-algebras.*

Proof. Immediate from the lemmas. □

References

[1] Kowalski, T., Słomczyńska, K.: Quasivarieties of p-algebras: Some new results. Studia Logica (2025). DOI 10.1007/s11225-025-10187-9. URL https://doi.org/10.1007/s11225-025-10187-9